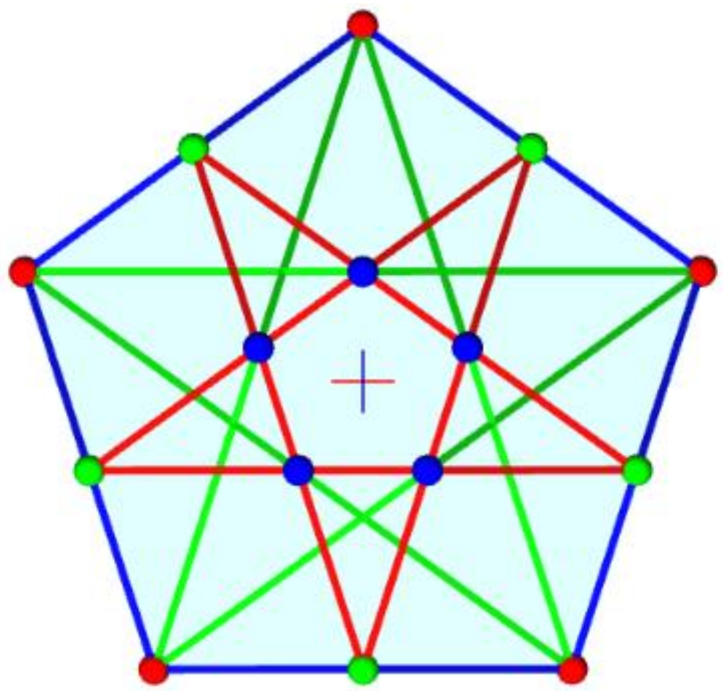
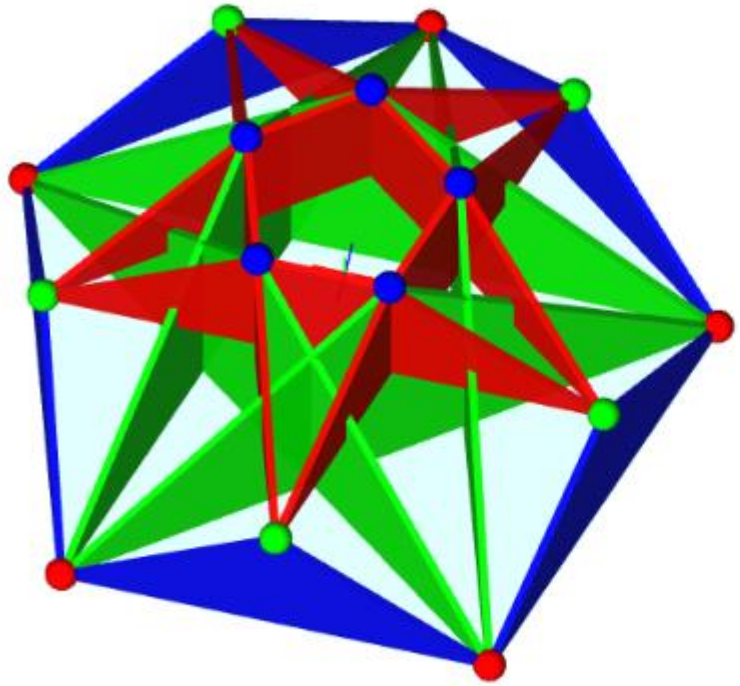
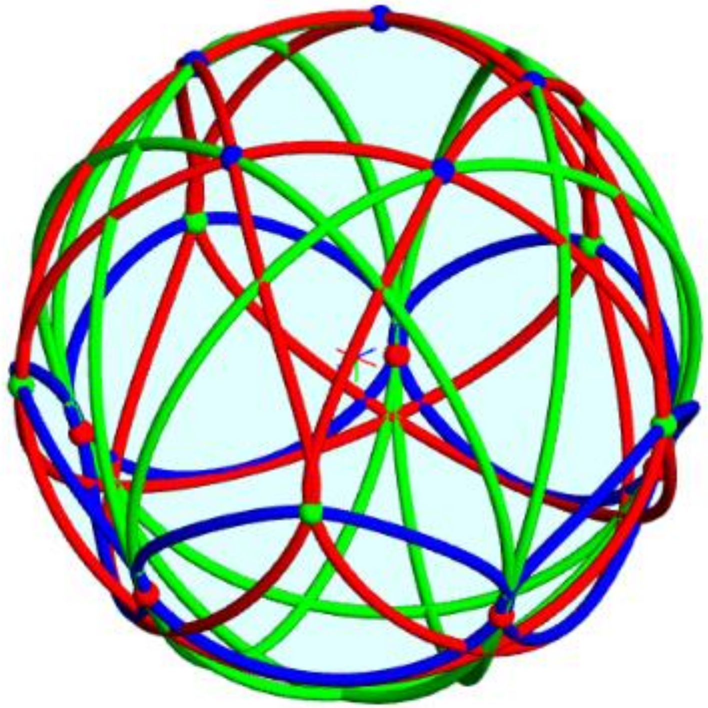
Introduction to Hypergons

What is a Hypergon?

A **hypergon** is a graph (or hypergraph), a network of vertices and edges.

A **regular hypergon** is (abstractly) a symmetric hypergraph (a graph if *k*=2)

It can also be a Point-Line Configuration (v_d e_k) with vd=ek

The **incidence matrix** of a regular hypergon is:

$\begin{bmatrix} v & d \\ k & e \end{bmatrix}$, with *v* vertices, *d* edges/vertex
k vertices/edge, *e* edges.

Ordinary ***polygons*** are (p_2) $\begin{bmatrix} p & 2 \\ 2 & p \end{bmatrix}$ (dih(*p*), dihedral symmetry)

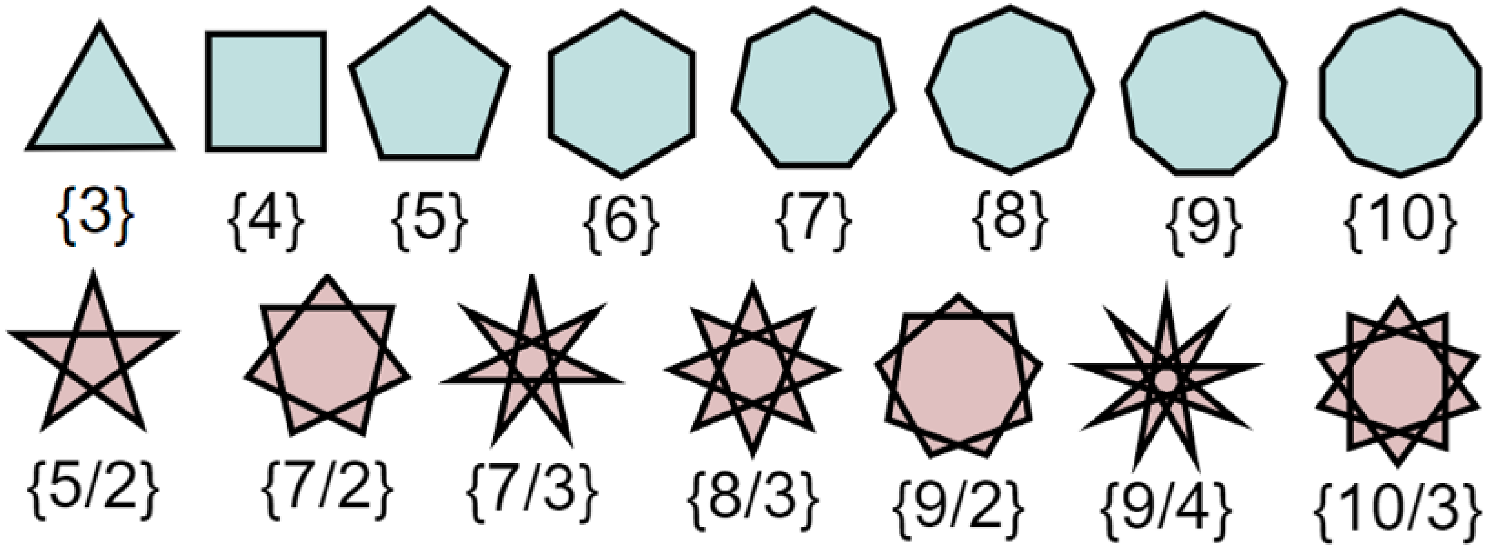
Example: polygons (2-regular graph)

A **regular polygon**, {*p*} has *p* vertex, *p* edges, with 2 vertices per edge, and 2 edges per vertex. A **regular polygram** is {*p*/*q*}, with gcd{*p*,*q*}=1.

{5} is a **pentagon**, {5/2} is a **pentagram**.

A *p*-gon or *p*/*q*-gram has **incidence matrix**: $\begin{bmatrix} p & 2 \\ 2 & p \end{bmatrix}$

Configuration: $(p_2\, p_2) = (p_2)$

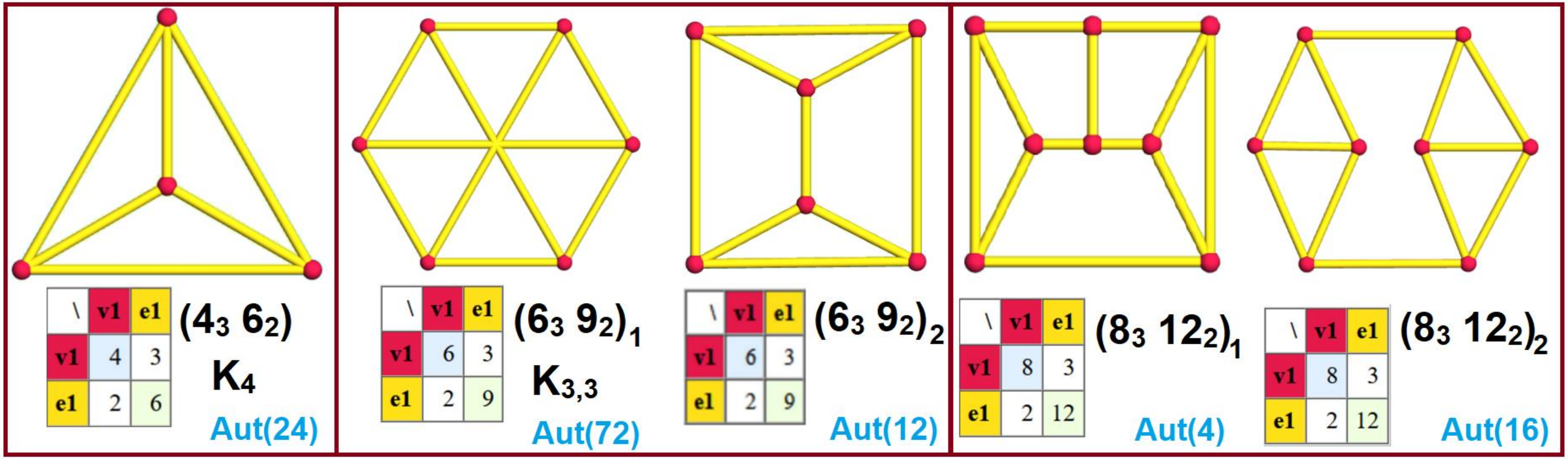


Example: 3-regular (cubic) graphs

A cubic graph has vertices and edges, with 3 edges/vertex.

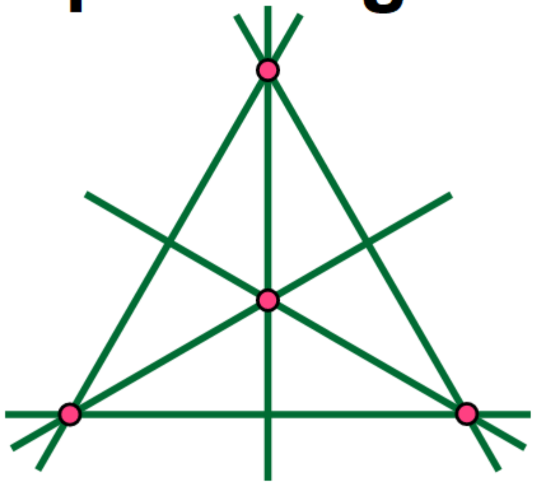
It can be seen as point-line configuration $(v_3\ e_2)$, with 3v=2e.

Some can be skeletons of 3D polyhedra, like K_4 is skeleton of tetrahedron.



Example: point-line configurations in plane

Complete quadrangle

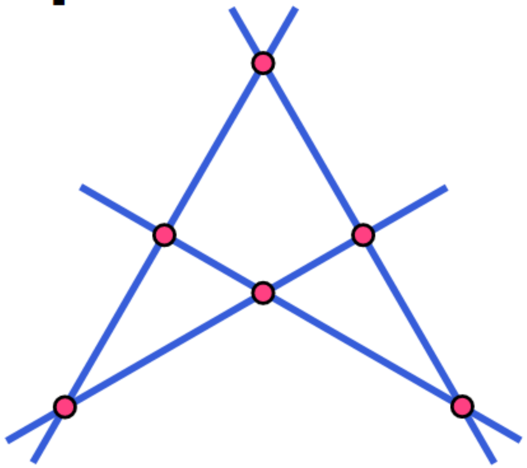


\	v1	e1
v1	4	3
e1	2	6

$(4_3\ 6_2)$

Aut(24)

Complete quadrilateral



\	v1	e1
v1	6	2
e1	3	4

$(6_2\ 4_3)$

Aut(24)

*A **configuration** in the plane consists of a finite set of points (v), and a finite arrangement of lines (e), such that each point is incident to the same number of lines (d) and each line is incident to the same number of points (k).*

The same symbol ($v_d\ e_k$) need not be isomorphic as incidence structures.

Incidence matrix $\begin{bmatrix} v & d \\ k & e \end{bmatrix}$

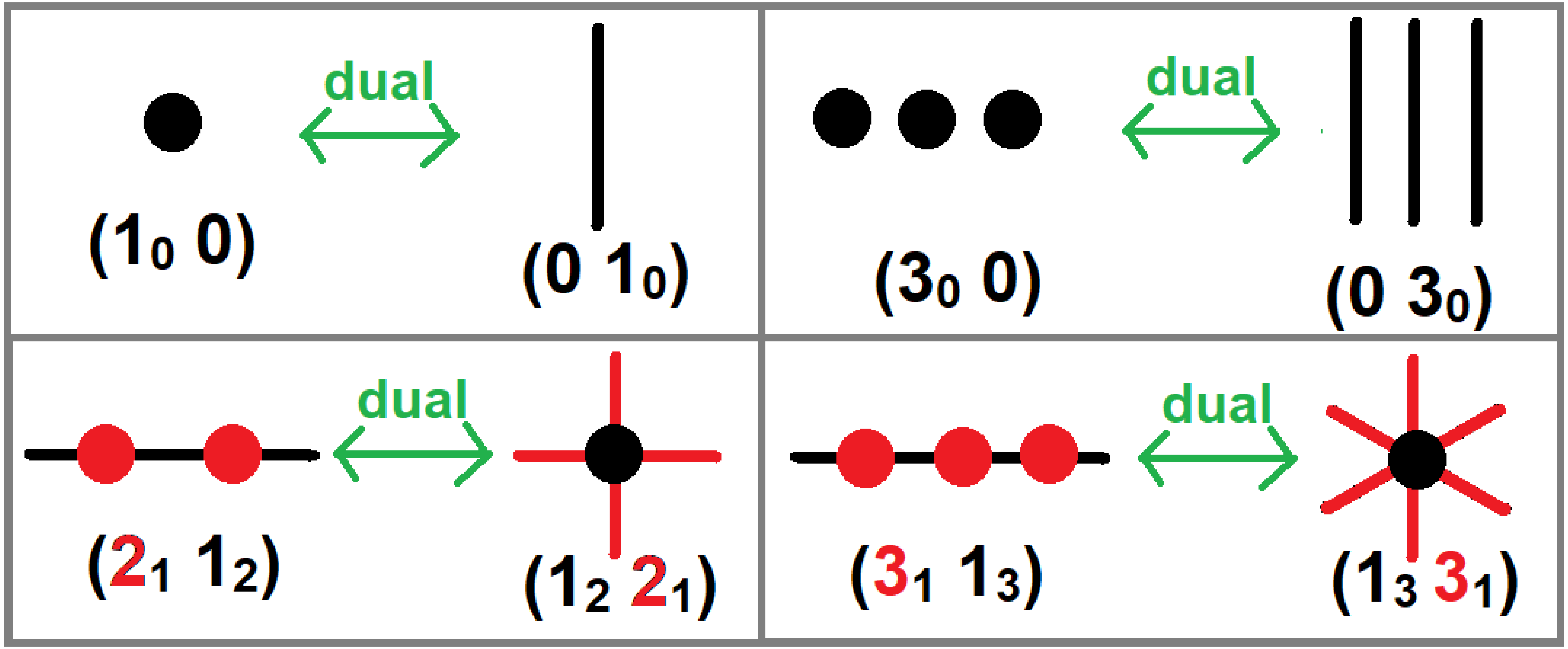
Dual Configuration swap points and lines. ($v_d\ e_k$) ←→ ($e_k\ v_d$)

Trivial configurations

Dual forms (n_0 0) and (0 n_0) are isolated points or lines

Dual form (n_1 1_n) and (1_n n_1) have v points on 1 line, or e lines with 1 point

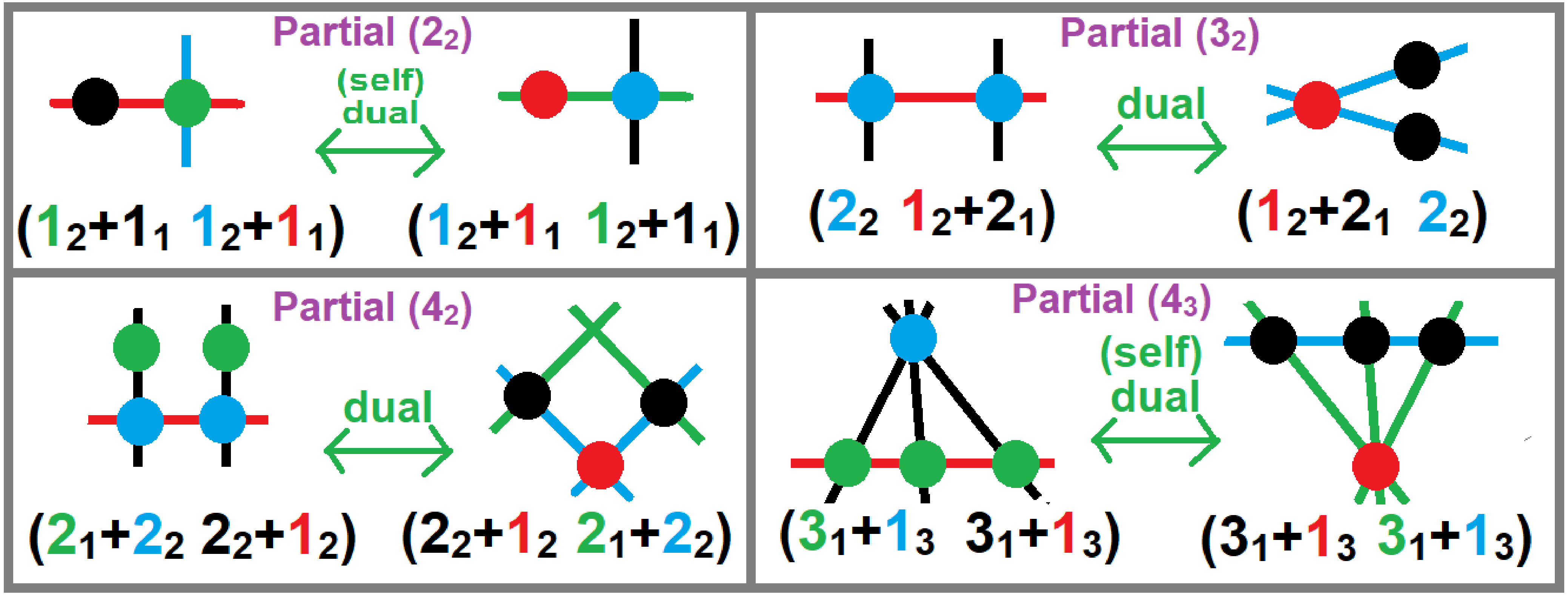
(1_1 1_1) is self-dual as 1 point incident to 1 line.



Partial configurations

A *partial configuration* allows some points and lines to have a lower incidence.

For example (3_2) has 3 points and 3 lines. If one line is removed, it becomes $(1_2{+}2_1\ 2_2)$, and if one point is removed, it becomes $(2_2\ 1_2{+}2_1)$

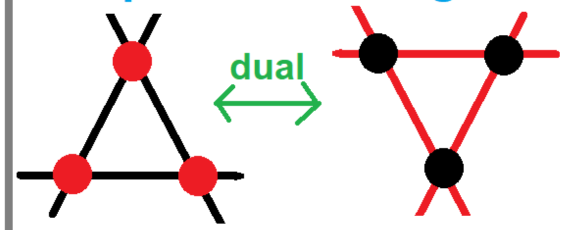


Element transitivity on a triangle

A triangle configuration (3_2) can have lower symmetry forms, using color to represent transitivity classes. An equilateral triangle is 1,1-transitive. An isosceles triangle is 2,2-transitive, and a scalene triangle 3,3-transitive.

The incidence matrix for a,b-transitive form to be (a+b)x(a+b) with element counts on the diagonal, and incidence off diagonal.

Equilateral triangle



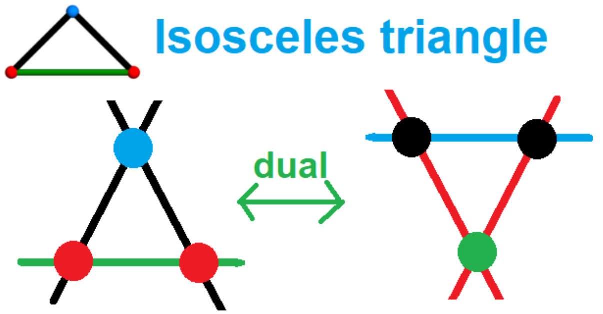
(3_2 3_2)

\	v1	e1
v1	3	2
e1	2	3

(3_2 3_2)

\	v1	e1
v1	3	2
e1	2	3

Isosceles triangle



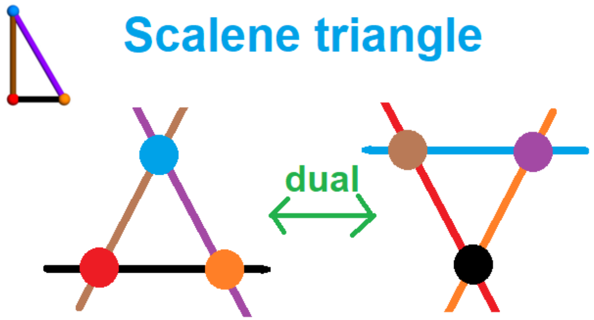
(2_2+1_2 2_2+1_2)

\	v1	v2	e1	e2
v1	2	*	1	1
v2	*	1	0	2
e1	2	0	1	*
e2	1	1	*	2

(2_2+1_2 2_2+1_2)

\	v1	v2	e1	e2
v1	1	*	2	0
v2	*	2	1	1
e1	1	1	2	*
e2	0	2	*	1

Scalene triangle



($1_2+1_2+1_2$ $1_2+1_2+1_2$)

\	v1	v2	v3	e1	e2	e3
v1	1	*	*	1	0	1
v2	*	1	*	1	1	0
v3	*	*	1	0	1	1
e1	1	1	0	1	*	*
e2	0	1	1	*	1	*
e3	1	0	1	*	*	1

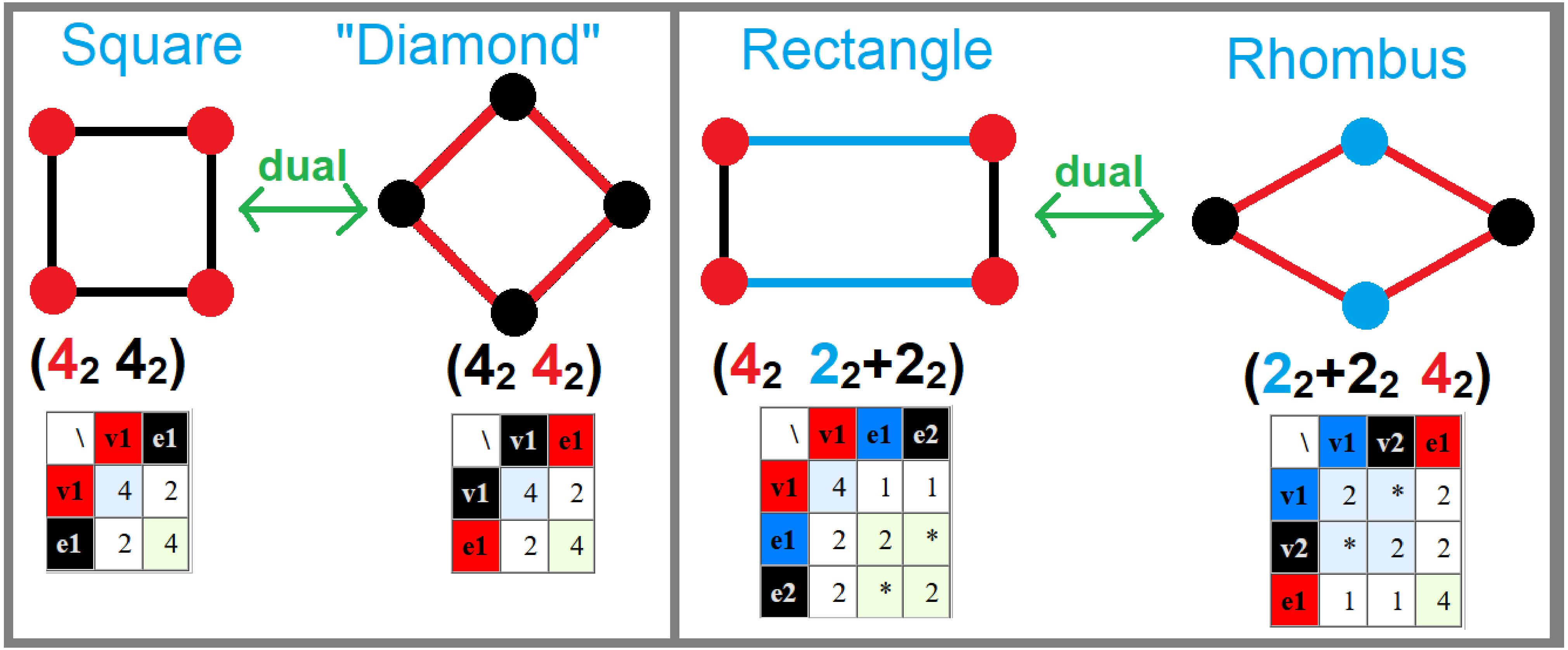
($1_2+1_2+1_2$ $1_2+1_2+1_2$)

\	v1	v2	v3	e1	e2	e3
v1	1	*	*	1	1	0
v2	*	1	*	0	1	1
v3	*	*	1	1	0	1
e1	1	0	1	1	*	*
e2	1	1	0	*	1	*
e3	0	1	1	*	*	1

Element transitivity on a square

A square configuration (4_2) also has lower symmetry forms.

A square is 1,1-transitive. Others include a rectangle is 1,2-transitive, and its dual rhombus is 2,1-transitive.



Square:

\	v1	e1
v1	4	2
e1	2	4

"Diamond":

\	v1	e1
v1	4	2
e1	2	4

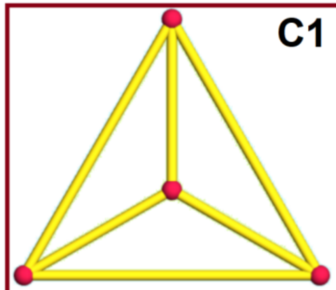
Rectangle:

\	v1	e1	e2
v1	4	1	1
e1	2	2	*
e2	2	*	2

Rhombus:

\	v1	v2	e1
v1	2	*	2
v2	*	2	2
e1	1	1	4

Cubic graphs with a,b-transitivity by color C1, C2, C3, C4, C6

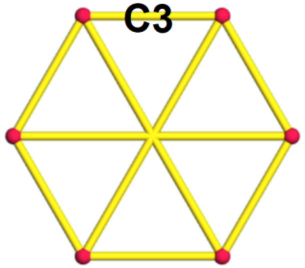


\	v1	e1
v1	4	3
e1	2	6

$(4_3\ 6_2)$

K_4

Aut(24)

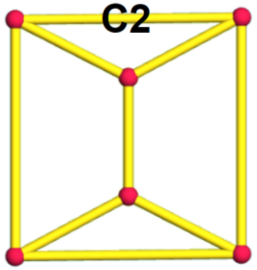


\	v1	e1
v1	6	3
e1	2	9

$(6_3\ 9_2)_1$

$K_{3,3}$

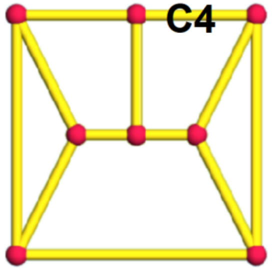
Aut(72)



\	v1	e1
v1	6	3
e1	2	9

$(6_3\ 9_2)_2$

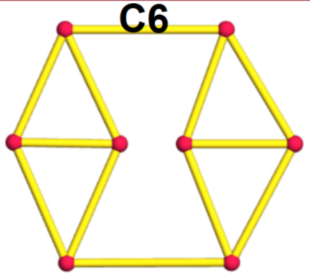
Aut(12)



\	v1	e1
v1	8	3
e1	2	12

$(8_3\ 12_2)_1$

Aut(4)

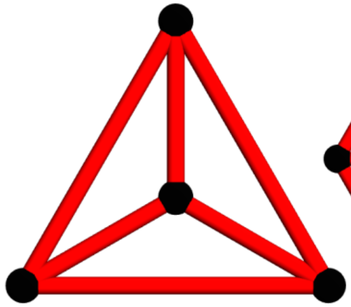


\	v1	e1
v1	8	3
e1	2	12

$(8_3\ 12_2)_2$

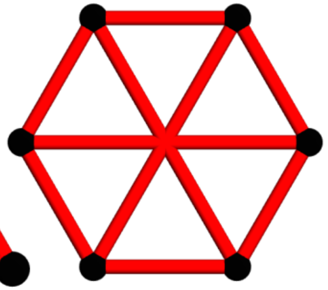
Aut(16)

1,1-transitive



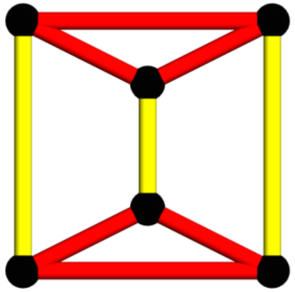
\	v1	e1
v1	4	3
e1	2	6

1,1-transitive



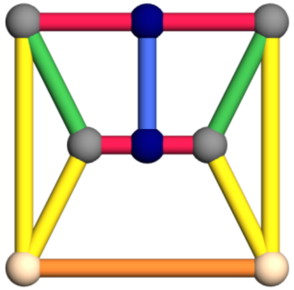
\	v1	e1
v1	6	3
e1	2	9

1,2-transitive



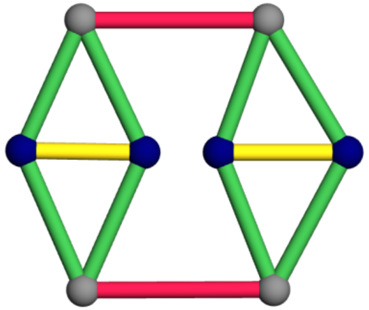
\	v1	e1	e2
v1	6	2	1
e1	2	6	*
e2	2	*	3

3,5-transitive



\	v1	v2	v3	e1	e2	e3	e4	e5
v1	4	*	*	1	0	1	0	1
v2	*	2	*	0	1	2	0	0
v3	*	*	2	0	0	0	1	2
e1	2	0	0	2	*	*	*	*
e2	0	2	0	*	1	*	*	*
e3	1	1	0	*	*	4	*	*
e4	0	0	2	*	*	*	1	*
e5	1	0	1	*	*	*	*	4

2,3-transitive



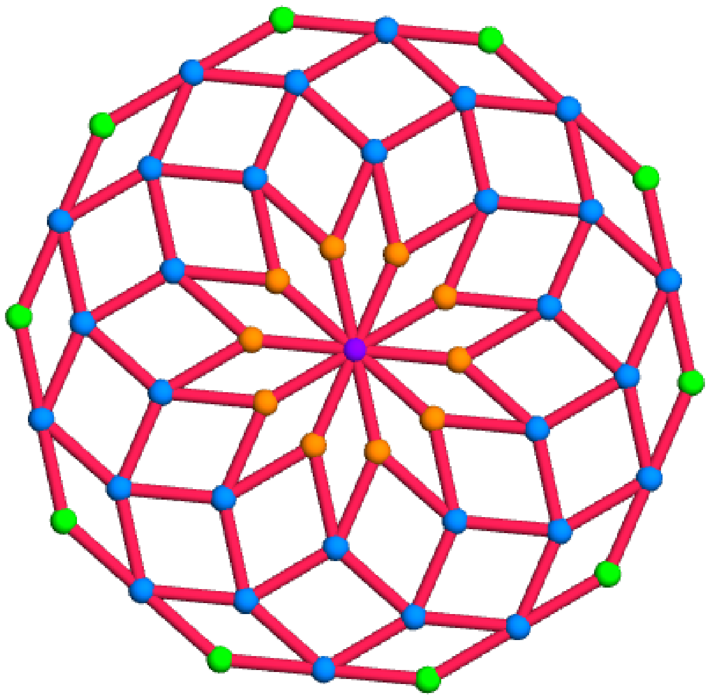
\	v1	v2	e1	e2	e3
v1	4	*	2	1	0
v2	*	4	2	0	1
e1	1	1	8	*	*
e2	2	0	*	2	*
e3	0	2	*	*	2

Decagonal rhombic rosette

Example – A rhombic rosette has a skeleton graph with 51 vertices and 90 edges. Its vertices have degrees 2, 3, 4, and 10.

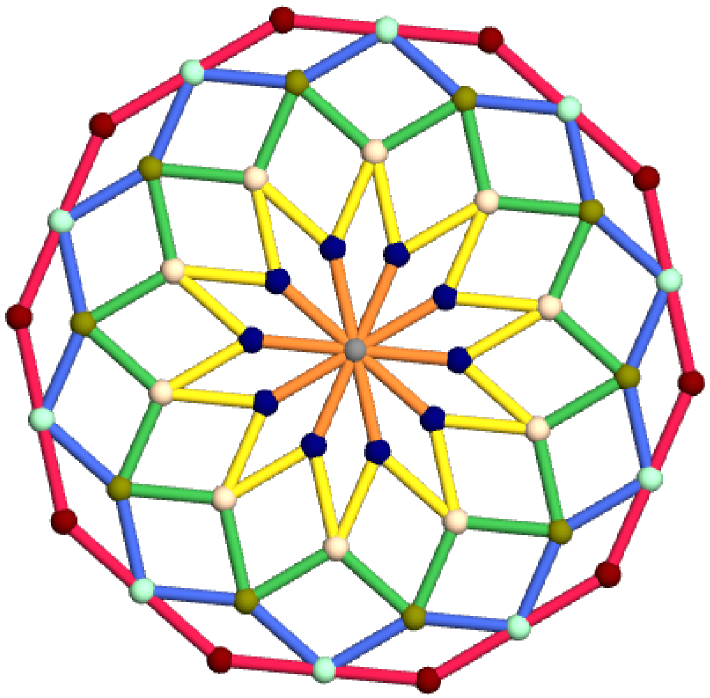
By transitivity, it has 6 vertex classes and 5 edges, seen in the planar symmetry.

Degree-colored



$(1_{10}+10_3+30_4+10_2\ \ 90_2)$

6,5-transitive colored



\	v1	v2	v3	v4	v5	v6	e1	e2	e3	e4	e5
v1	1	*	*	*	*	*	0	0	0	0	10
v2	*	10	*	*	*	*	0	0	2	0	1
v3	*	*	10	*	*	*	0	2	2	0	0
v4	*	*	*	10	*	*	0	2	0	2	0
v5	*	*	*	*	10	*	2	0	0	2	0
v6	*	*	*	*	*	10	2	0	0	0	0
e1	0	0	0	0	1	1	20	*	*	*	*
e2	0	0	1	1	0	0	*	20	*	*	*
e3	0	1	1	0	0	0	*	*	20	*	*
e4	0	0	0	1	1	0	*	*	*	20	*
e5	1	1	0	0	0	0	*	*	*	*	10

Self dual partial configuration

Branko Grünbaum (2009) shows a self-dual partial configuration with 15 point and 15 lines. (5_4+10_3) or (5_4+5_3+5_3) with 3 transitivity classes.

The 3,3-transitive structure has incidence matrix on the right.

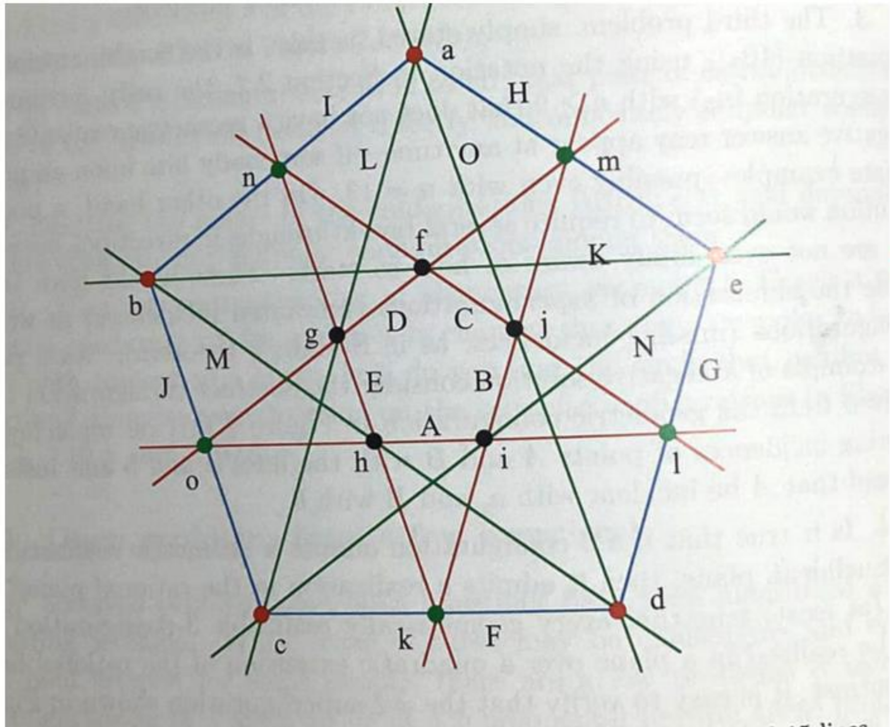
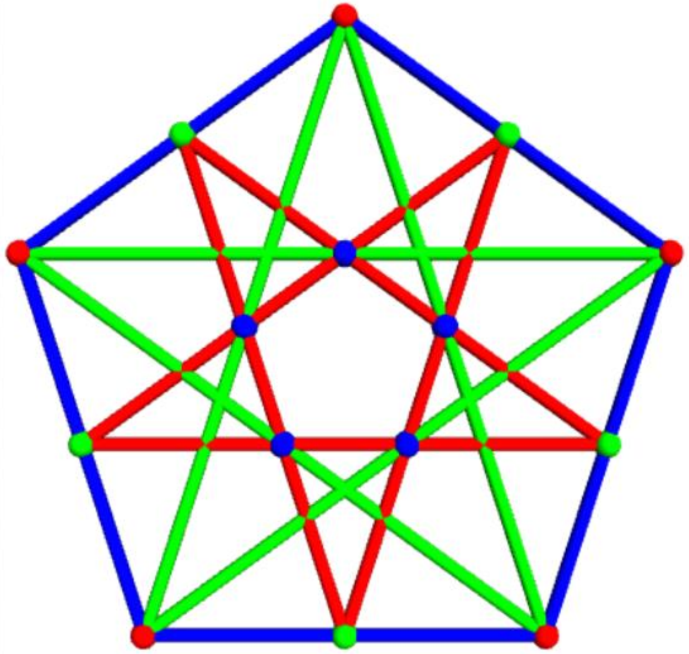


Figure 2.11.2. An intriguing selfdual collection of 15 points and 15 lines.



$(5_4+5_3+5_3)$

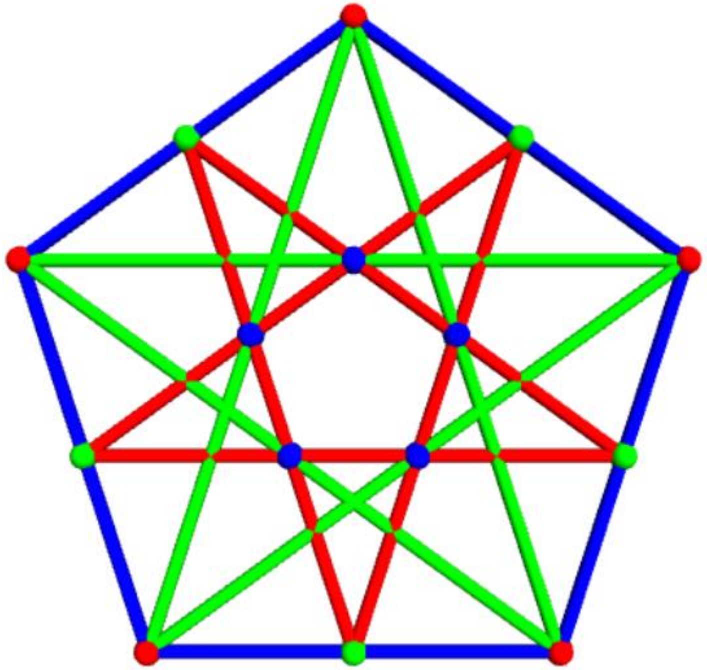
\	v1	v2	v3	e1	e2	e3
v1	5	*	*	2	2	0
v2	*	5	*	1	0	2
v3	*	*	5	0	1	2
e1	2	1	0	5	*	*
e2	2	0	1	*	5	*
e3	0	2	2	*	*	5

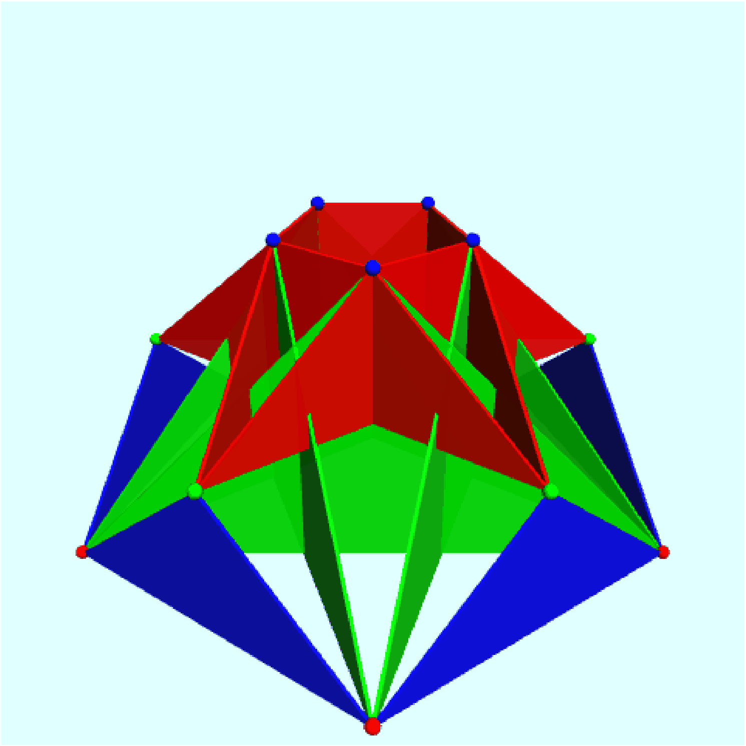
Hypergons - representations beyond the plane!

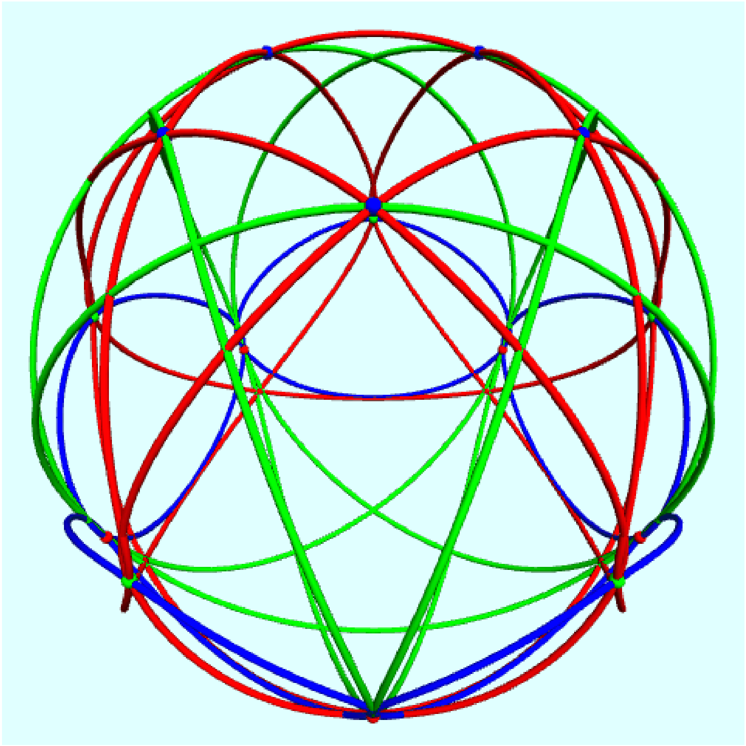
Extend planar points into z on transitivity class. [Config-15]

Draw 3-point lines as triangles, 4-point lines as quadrilateral

Draw “circles” (cubic spline loops) that pass through the points







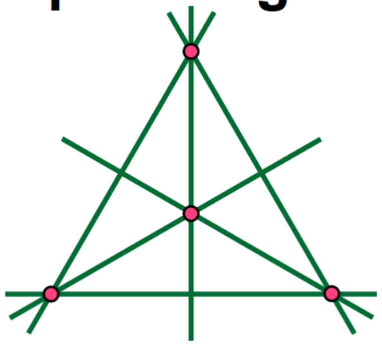
Complete Quadrangle and Quadrilateral

The complete quadangle has 4 vertices, 6 edges, on tetrahedral skeleton.

The complete quadrilateral has 6 vertices, 4 edges, seen as alternate triangles on an octahedron. (Or 4 tangent circles!)

Duality expressed by new vertices centered on original “edges”.

Complete quadrangle

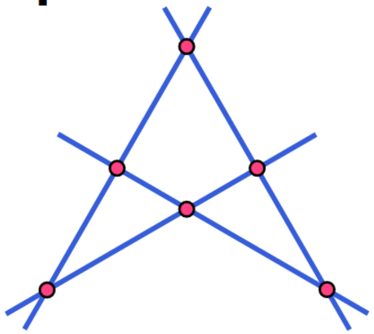


\	v1	e1
v1	4	3
e1	2	6

$(4_3\ 6_2)$

Aut(24)

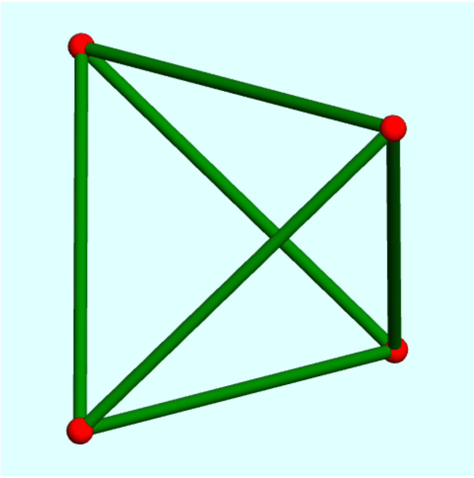
Complete quadrilateral

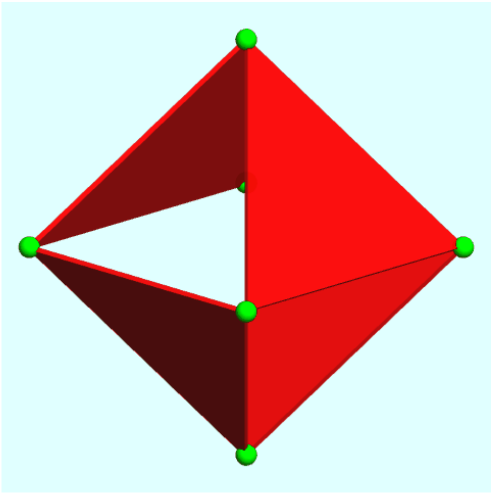


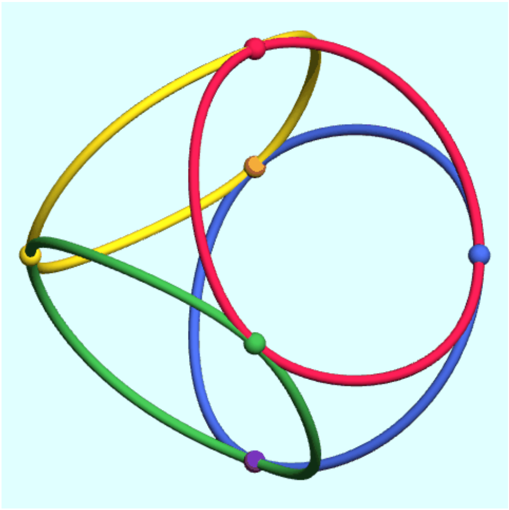
\	v1	e1
v1	6	2
e1	3	4

$(6_2\ 4_3)$

Aut(24)

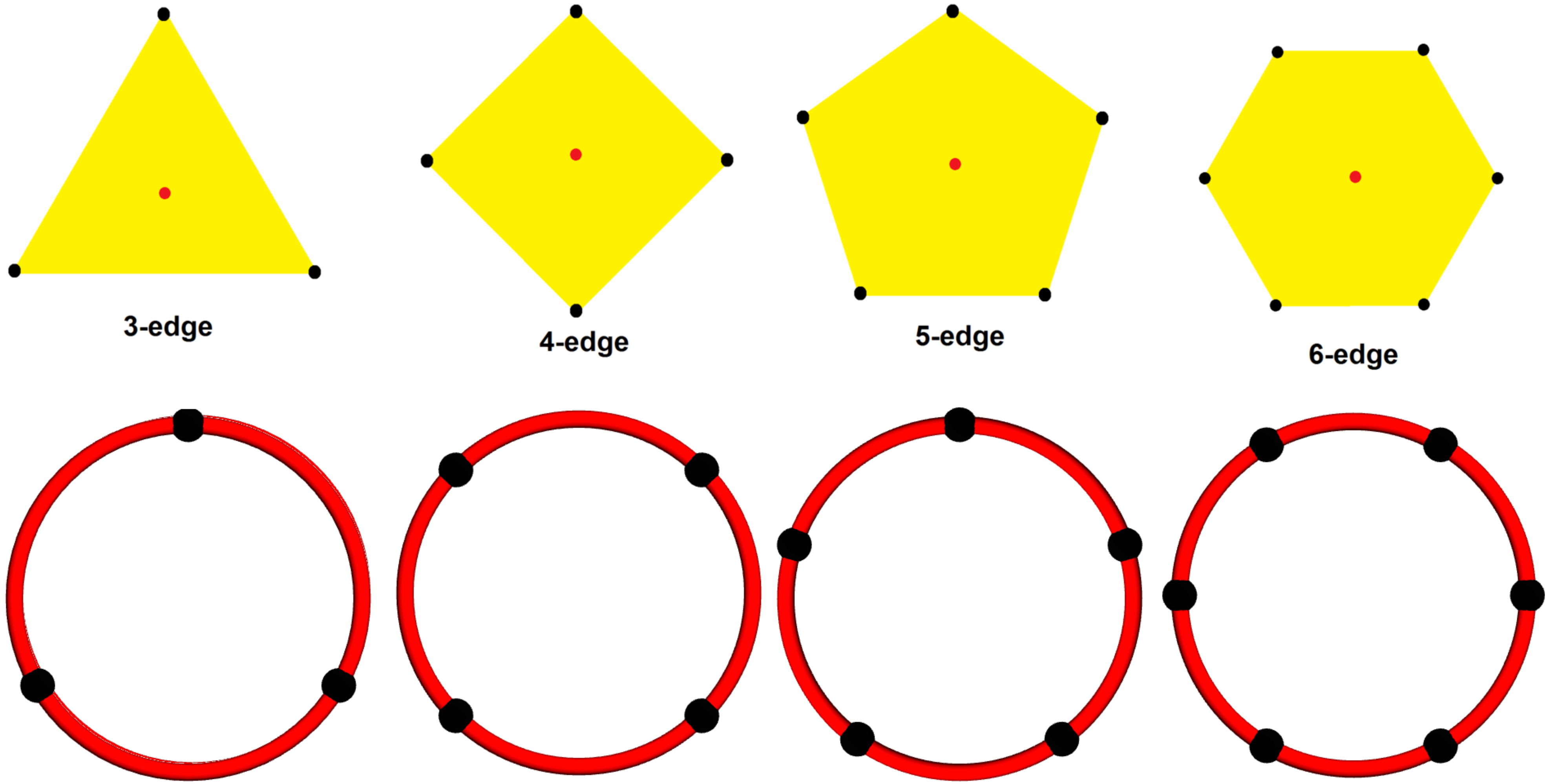






Representing hyperedges (3 or more vertices)

Edges with 3 or more vertices in a plane can be represented as a solid polygon or a circle (loop)

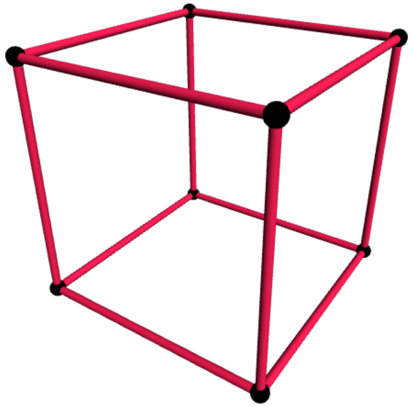


Cube as a hypergon (vertices and hyperedges)

A cube is a polyhedron with 8 vertices, 12 edges, and 6 square faces.

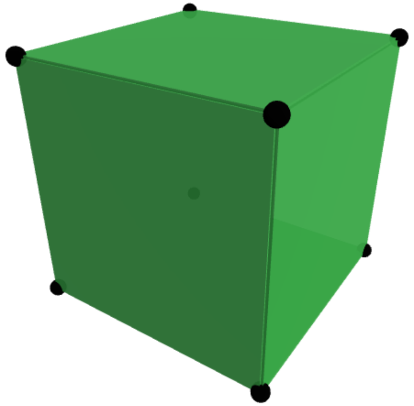
It can be reduced to a skeleton (graph) (8_3 12_2), or make 4-vertex edges from the square faces (8_3 6_4), Miquel config, or both (8_{3+3} 12_2+6_4)

Cube skeleton



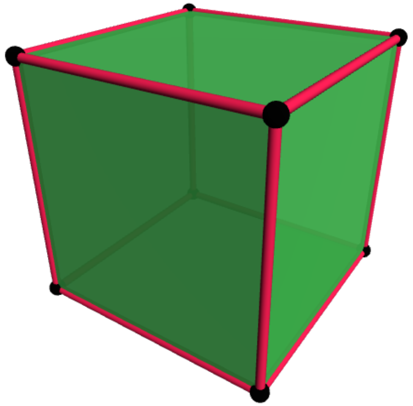
\	vl	el
vl	8	3
el	2	12

Square-edges

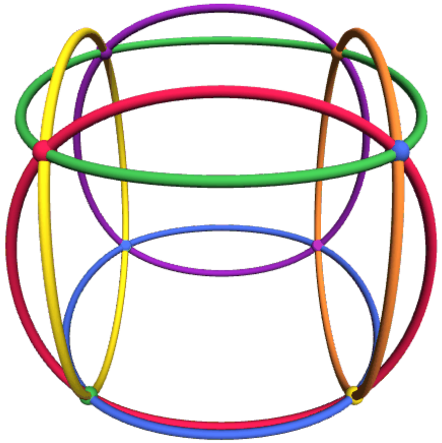


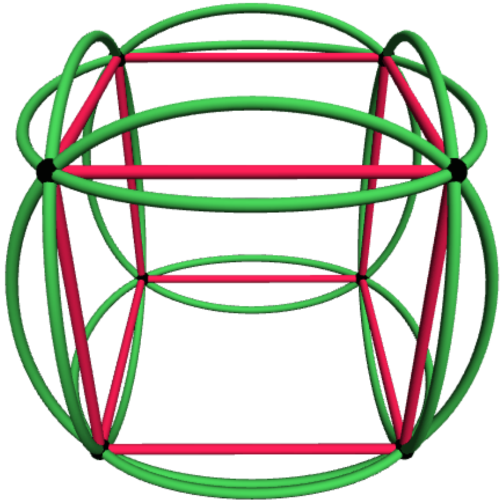
\	vl	el
vl	8	3
el	4	6

Rank 3



\	vl	el	e2
vl	8	3	3
el	2	12	2
e2	4	4	6

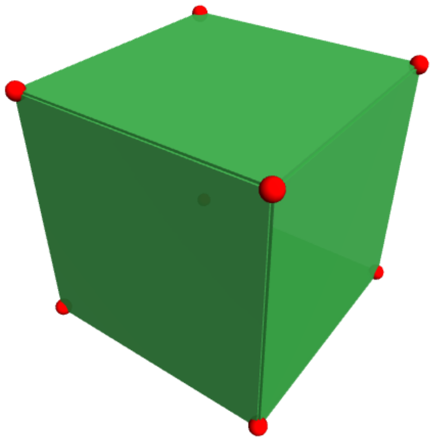


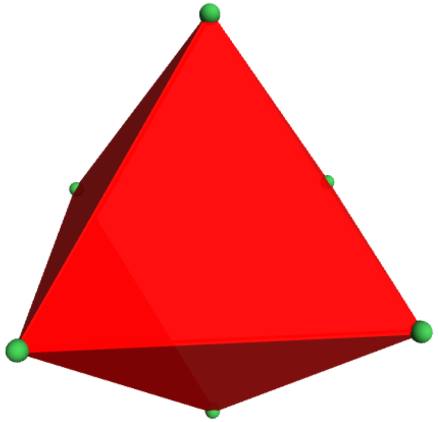


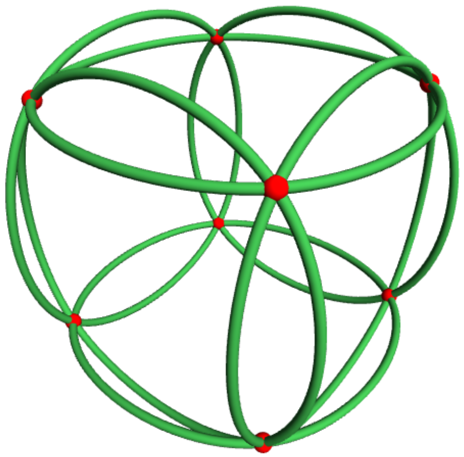
Cube-octahedron duality

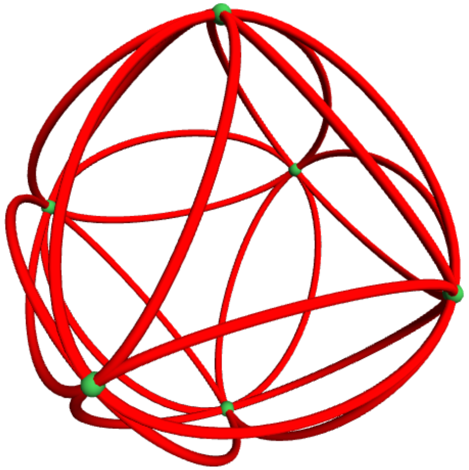
The cube and octahedron are dual polyhedra. They are also dual as hypergons $(8_3\ 6_4)$ and $(6_4\ 8_3)$

Cube hypergon **Octahedral hypergon**









\	vl	el
vl	8	3
el	4	6

\	vl	el
vl	6	4
el	3	8

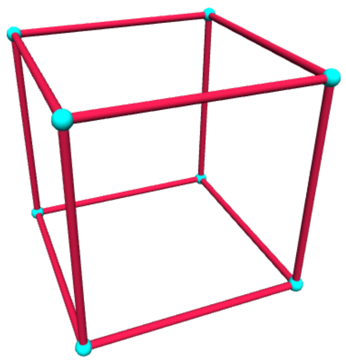
Cube skeleton truncated

A cube skeleton has 8 vertices and 12 edges.

The truncation makes new triangular edges at the vertices.

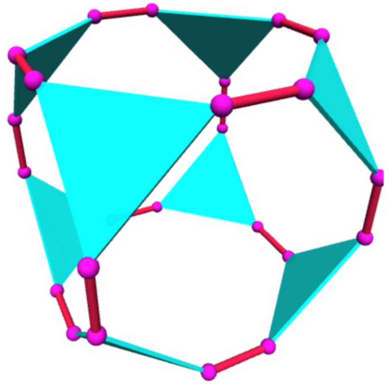
The dual of truncation is a “star” operator, and Levi graph

Cube skeleton



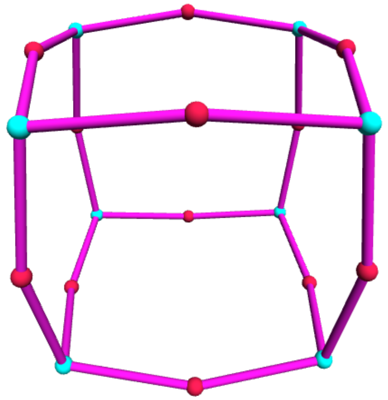
\	v1	e1
v1	8	3
e1	2	12

Truncated

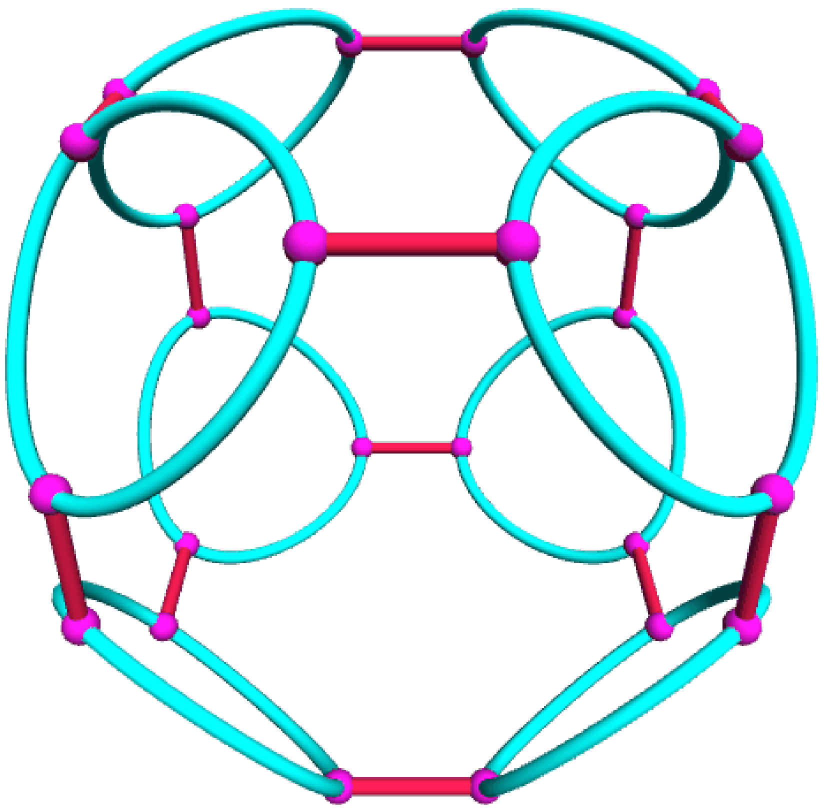


\	v1	e1	e2
v1	24	1	1
e1	2	12	*
e2	3	*	8

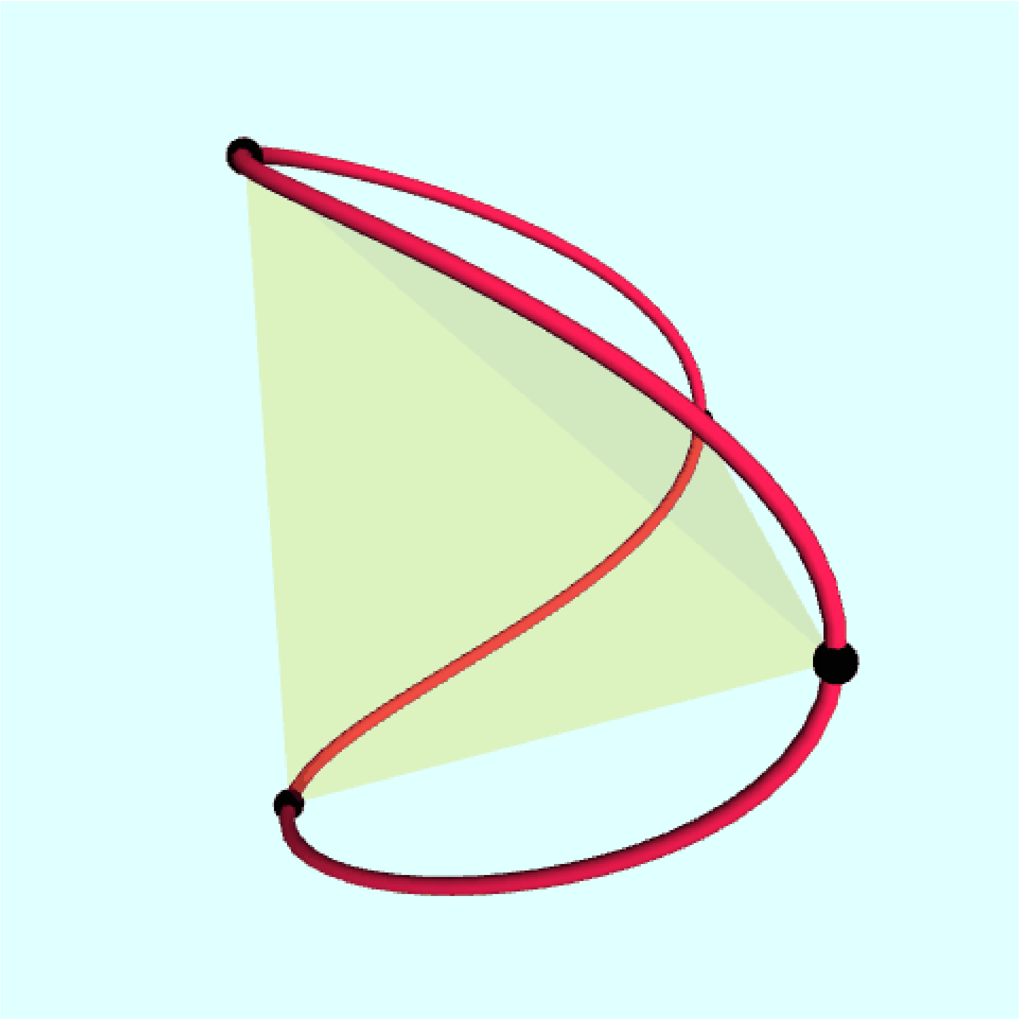
Dual



\	v1	v2	e1
v1	12	*	2
v2	*	8	3
e1	1	1	24



Nonplanar edges: convex hull and loop



Nonplanar edges

1)A convex hull

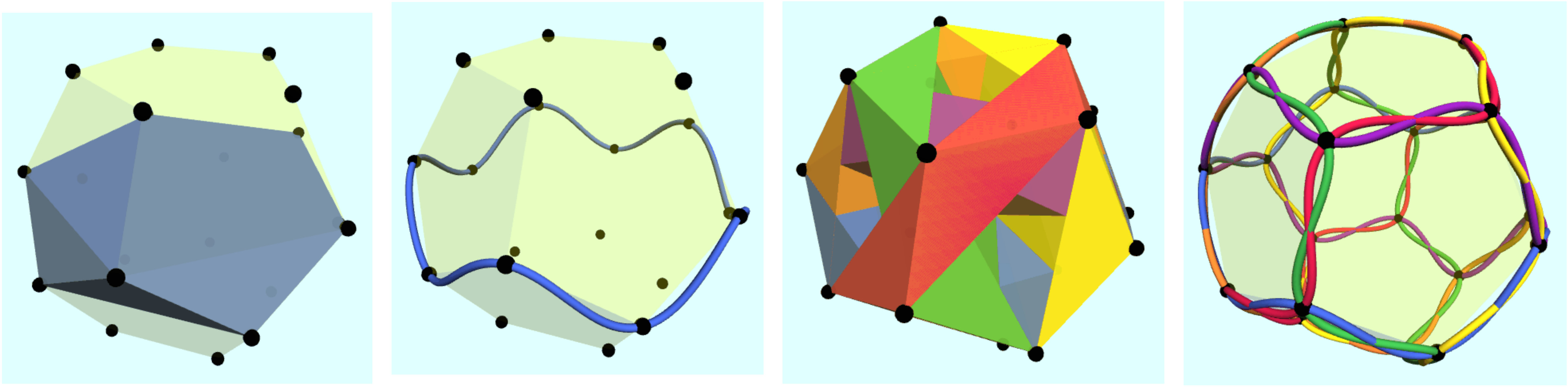
2)A cubic spline loop

Petrial dodecahedron hypergon

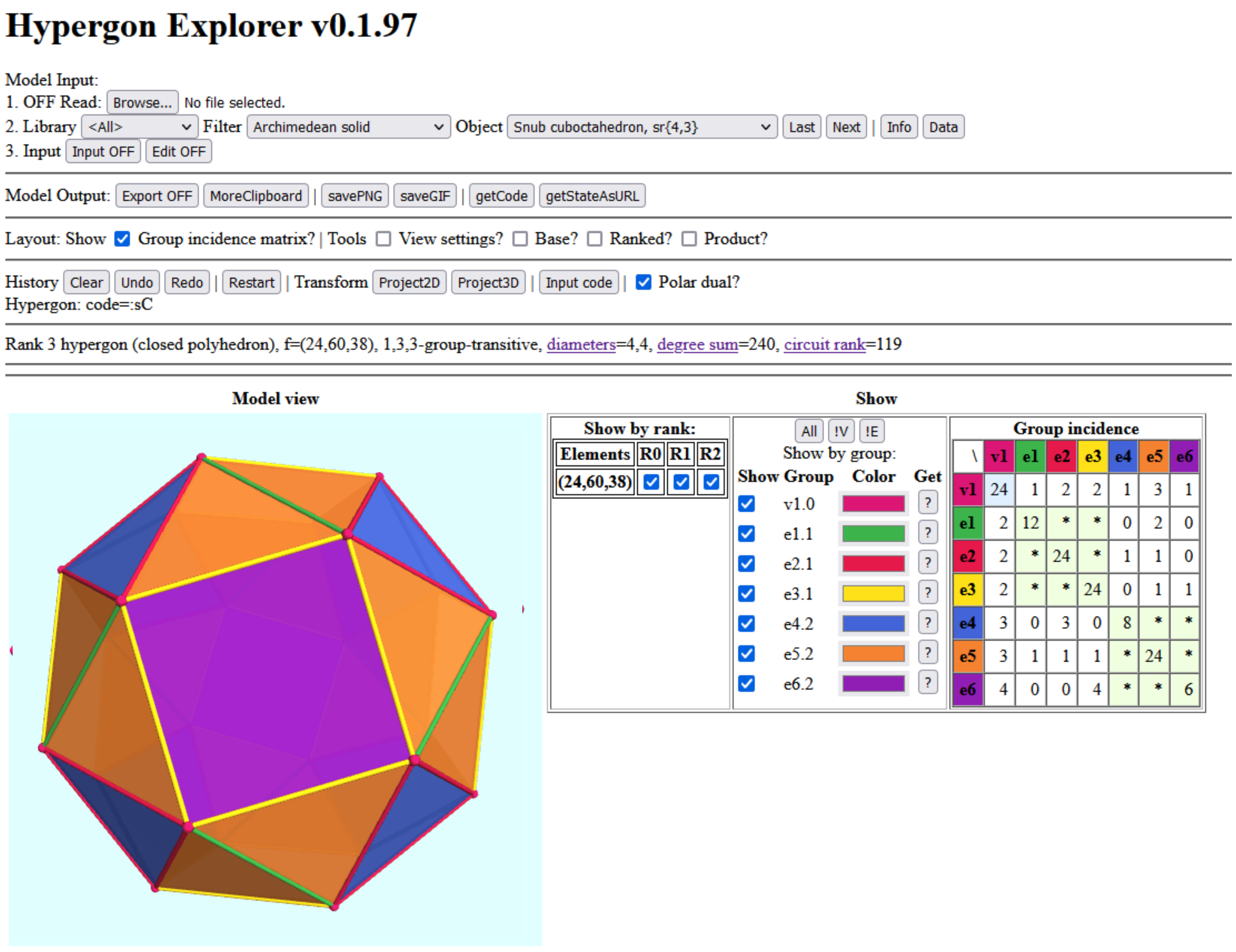
The 30 vertices of a dodecahedral skeleton (20_3 6_{10}) can be connected with 6 central skew decagonal edges.

Each edge can be drawn as a convex hull (pentagonal antiprism)

OR as a cubic spline path around axis on shortest dimension.



http://roice3.org/ruen/hyper/hypergon.html?code=sC



Hypergon explorer is an experimental javascript program on the web with a library of geometric shapes, with view options and operator to manipulate.

Example: Snub cube

Summary

A hypergon is a generalization of a polygon which allows 2 or more edges per vertex, and 2 or more vertices per edge.

- Graphs of points and line segments (Limit 2 vertices/edge)
- Configurations of points and lines (Lines are edges with 2+ points)
- Skeletons of polytopes represent hypergons in higher dimensions.

Structurally hypergons are hypergraphs, generalizations of graphs. Elements exist in transitivity classes and can be summarized by incidence matrices.

Hyperedges can be represented geometrically with convex hull interiors, circle paths, or cubic spline paths.

Hypergons exist dual pairs that reverse vertices and hyperedges, or may be self-dual.

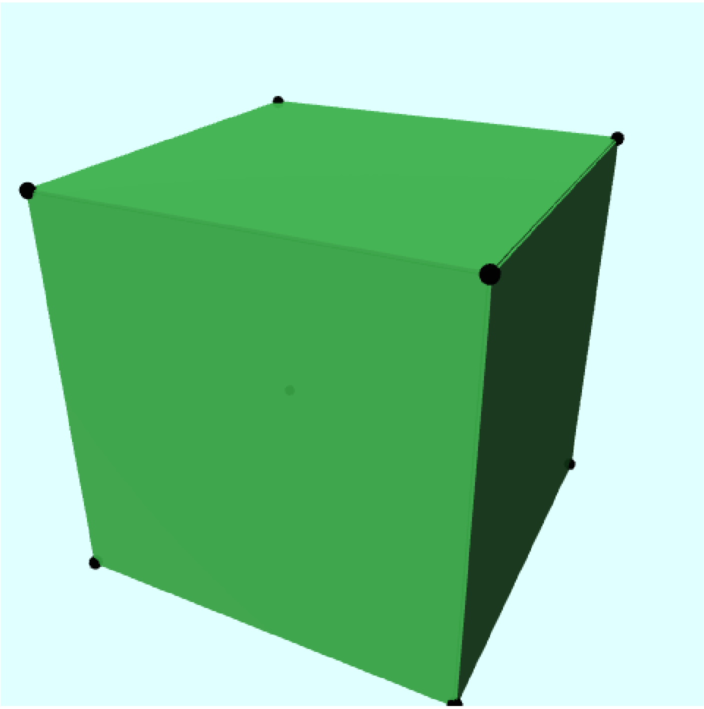
Ranked hypergons can represent multilevels of polytopes.

Hypergon format

The OFF file format defines polyhedra, listing vertices x,y,z first, and faces last.
Hypergons reinterpret faces as hyperedges.
Optional RGB color codes are used for grouping transitivity classes.

OFF # Cube
8 6
-1 1 1 **0 0 0**
-1 1 -1 0 0 0
1 1 -1 0 0 0
1 1 1 0 0 0
1 -1 1 0 0 0
-1 -1 1 0 0 0
-1 -1 -1 0 0 0
1 -1 -1 0 0 0
4 0 1 2 3 60 180 75
4 0 3 4 5 60 180 75
4 0 5 6 1 60 180 75
4 2 1 6 7 60 180 75
4 2 7 4 3 60 180 75
4 4 7 6 5 60 180 75

Model view



Show