CHAPTER V

POLYHEDRA

"Although a Discourse of Solid Bodies be an uncommon and neglected Part of Geometry, yet that it is no inconsiderable or unprofitable Improvement of the Science will (no doubt) be readily granted by such, whose Genius tends as well to the Practical as Speculative Parts of it, for whom this is chiefly intended."*

A polyhedron is a solid figure† with plane faces and straight edges, so arranged that every edge is both the join of two vertices and a common side of two faces. Familiar instances are the pyramids and prisms. (A pentagonal pyramid has six vertices, ten edges, and six faces; a pentagonal prism has ten, fifteen, and seven. See figure 7 on plate I.) I would mention also the *antiprism*,‡ whose two bases, though parallel, are not similarly situated, but each vertex of either corresponds to a side of the other, so that the lateral edges form a zigzag. (Thus a pentagonal antiprism has ten vertices, twenty edges, and twelve faces. See figure 9 on plate I.)

The tessellations described on page 106 may be regarded as infinite polyhedra.

SYMMETRY AND SYMMETRIES

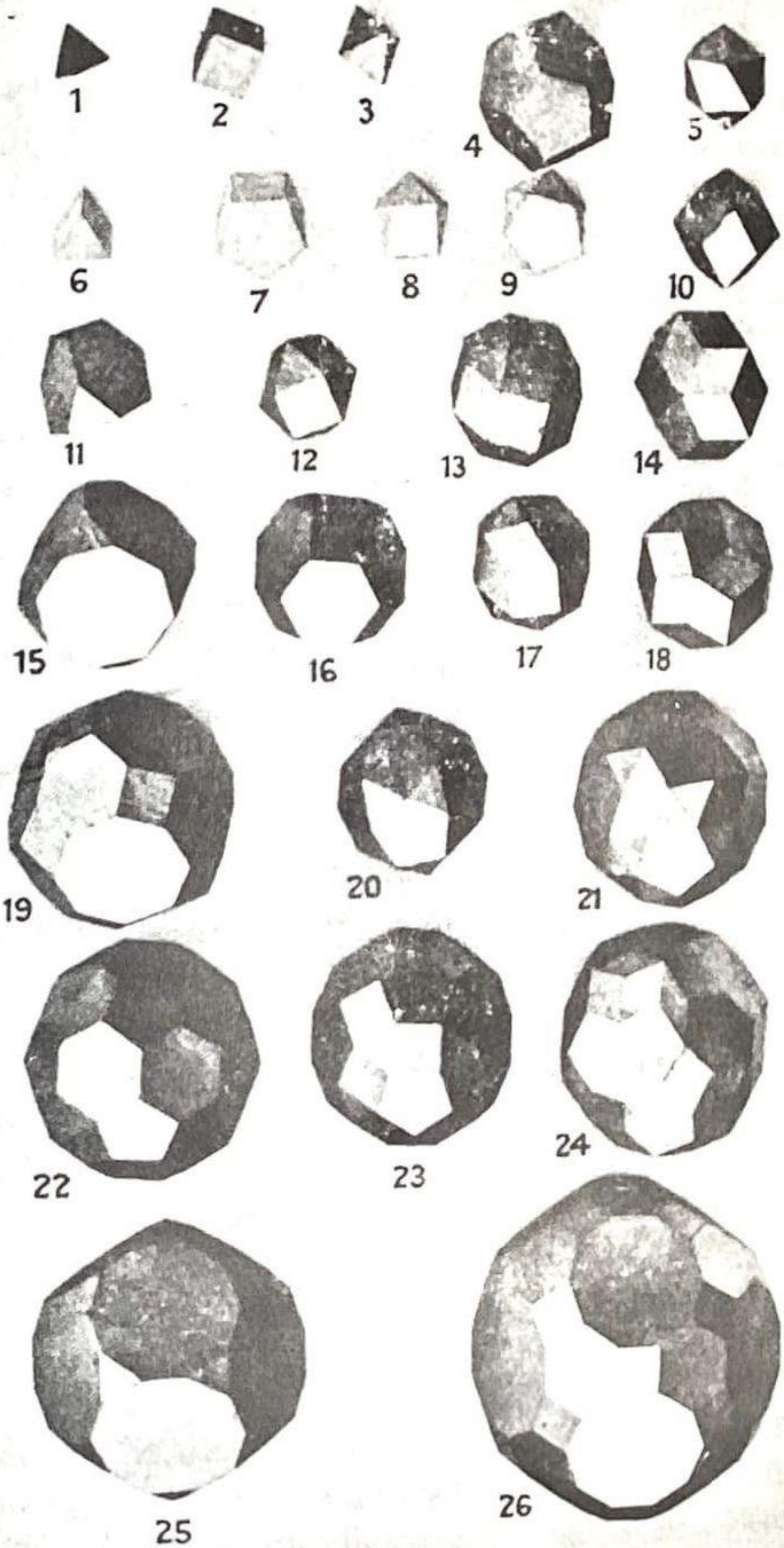
It is convenient to say that a figure is *reflexible*§ if it is superposable with its image in a plane mirror (i.e. if it is, in the most elementary sense, "symmetrical"). A figure which is not reflexible forms, with its mirror-image, an *enantiomorphous* pair.

*Abraham Sharp, *Geometry Improv'd*, London, 1717, p. 65.

†More precisely, it is the *surface* of such a solid figure.

‡Or "prismoid." See the *Encyclopaedia Britannica* (XIVth edition), article "Solids."

§Or "self-reflexible."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(The obvious example is a pair of shoes.) A reflexible figure has at least one *plane of symmetry*; the operation of reflecting in such a plane leaves the figure unchanged as a whole. A figure may also be symmetrical by rotation about an *axis of symmetry*. The vague statement that a figure has a certain amount of "symmetry" can be made precise by saying that the figure has a certain number of *symmetries*, a symmetry* being defined as any combination of motions and reflections which leaves the figure unchanged as a whole.

For a regular polygon $ABC\ldots X$, there is a symmetry (in fact, a rotation) which cyclically permutes the vertices, changing A into B, B into $C, \ldots,$ and X into A.

THE FIVE PLATONIC SOLIDS

Let A be a vertex belonging to a face α of a polyhedron. The polyhedron is said to be *regular* if it admits two particular symmetries: one which cyclically permutes the vertices of α, and one which cyclically permutes the faces surrounding A. It follows that all the faces are regular and equal, all the edges are equal, and all the vertices are surrounded alike. If each vertex is surrounded by q p-gons, we may denote the polyhedron by the symbol p^q (as on page 106) or $\{p, q\}$.

If the polyhedron is finite, the faces at one vertex form a solid angle. The internal angle of each face being $(p-2)\pi/p$, we now have $q(p-2)\pi/p<2\pi$, that is,

$$(p-2)(q-2)<4.$$

Since p and q are both greater than 2, we merely have to consider the possible ways of expressing 1 or 2 or 3 as the product of two positive integers, and then in each case the polyhedron can be built up, face by face. Letting V, E, F denote the number of vertices, edges, faces, the results are as in the following table. (See plate I, figure 1, 2, 3, 4, 5.)

*Or "symmetry operation." (Any rotation or translation may be regarded as a combination of two reflections.)

$\{p, q\}$	V	E	F	Name
$\{3, 3\}$	4	6	4	Regular tetrahedron
$\{4, 3\}$	8	12	6	Cube
$\{3, 4\}$	6	12	8	Octahedron
$\{5, 3\}$	20	30	12	Dodecahedron
$\{3, 5\}$	12	30	20	Icosahedron

Clearly, $qV = 2E = pF$. A less obvious relation is

$$E^{-1} = p^{-1} + q^{-1} - \tfrac{1}{2}.$$

This follows easily from Euler's Formula $F + V - E = 2$, which will be proved in chapter VIII, on pages 232–233.

In four ways the tetrahedron can be regarded as a triangular pyramid, and the octahedron as a triangular antiprism. In three ways the octahedron can be regarded as a square double-pyramid, and the cube as a square prism. In six ways the icosahedron can be regarded as a pentagonal antiprism with two pentagonal pyramids stuck on to its bases. The faces of the dodecahedron consist of two opposite pentagons (in parallel planes), each surrounded by five other pentagons.

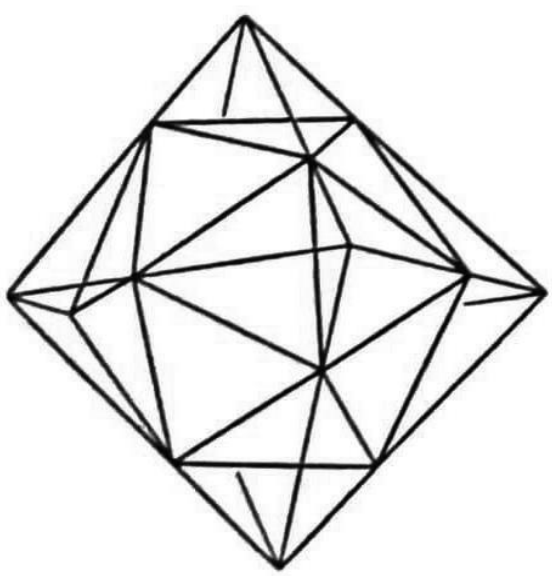
An icosahedron can be inscribed in an octahedron, so that each vertex of the icosahedron divides an edge of the octahedron according to the "golden section."* A cube can be inscribed in a dodecahedron so that each edge of the cube lies in a face of the dodecahedron (and joins two alternate vertices of that face).

These five figures have been known since ancient times. The earliest thorough investigation of them is probably that of Theaetetus.† It has been suggested that Euclid's *Elements* was originally written, not as a general treatise on geometry, but in order to supply the necessary steps for a full appreciation of the five regular solids. At any rate, Euclid begins by constructing an equilateral triangle, and ends by constructing a dodecahedron.

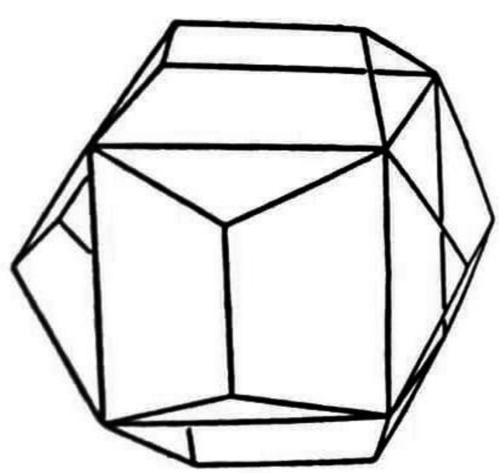
*See page 39.
†T. Heath, *A History of Greek Mathematics*, Oxford, 1921, vol. I, p. 162.

Mystically minded Greeks associated the regular polyhedra with the four Elements and the Universe. Kepler* justified the correspondence as follows. Of the five solids, the tetrahedron has the smallest volume for its surface, the icosahedron the largest; these therefore exhibit the qualities of dryness and wetness, respectively, and correspond to Fire and Water. The cube, standing firmly on its base, corresponds to the stable Earth; but the octahedron, which rotates freely when held by two opposite corners, corresponds to the mobile Air. Finally, the dodecahedron corresponds to the Universe, because the zodiac has twelve signs. He illustrated this correspondence by drawing a bonfire on his tetrahedron; a lobster and fishes on his icosahedron; a tree, a carrot, and gardening tools on his cube; birds and clouds on his octahedron; and the sun, moon, and stars on his dodecahedron.

With each of these polyhedra we may associate three concentric spheres: one (the "circum-sphere") through all the vertices, one touching all the edges, and one (the "insphere") touching all the faces. Consider the second of these spheres. If we replace each edge by a perpendicular line touching this sphere at the same point, the edges at a vertex lead to the sides



Icosahedron and Octahedron



Cube and Dodecahedron

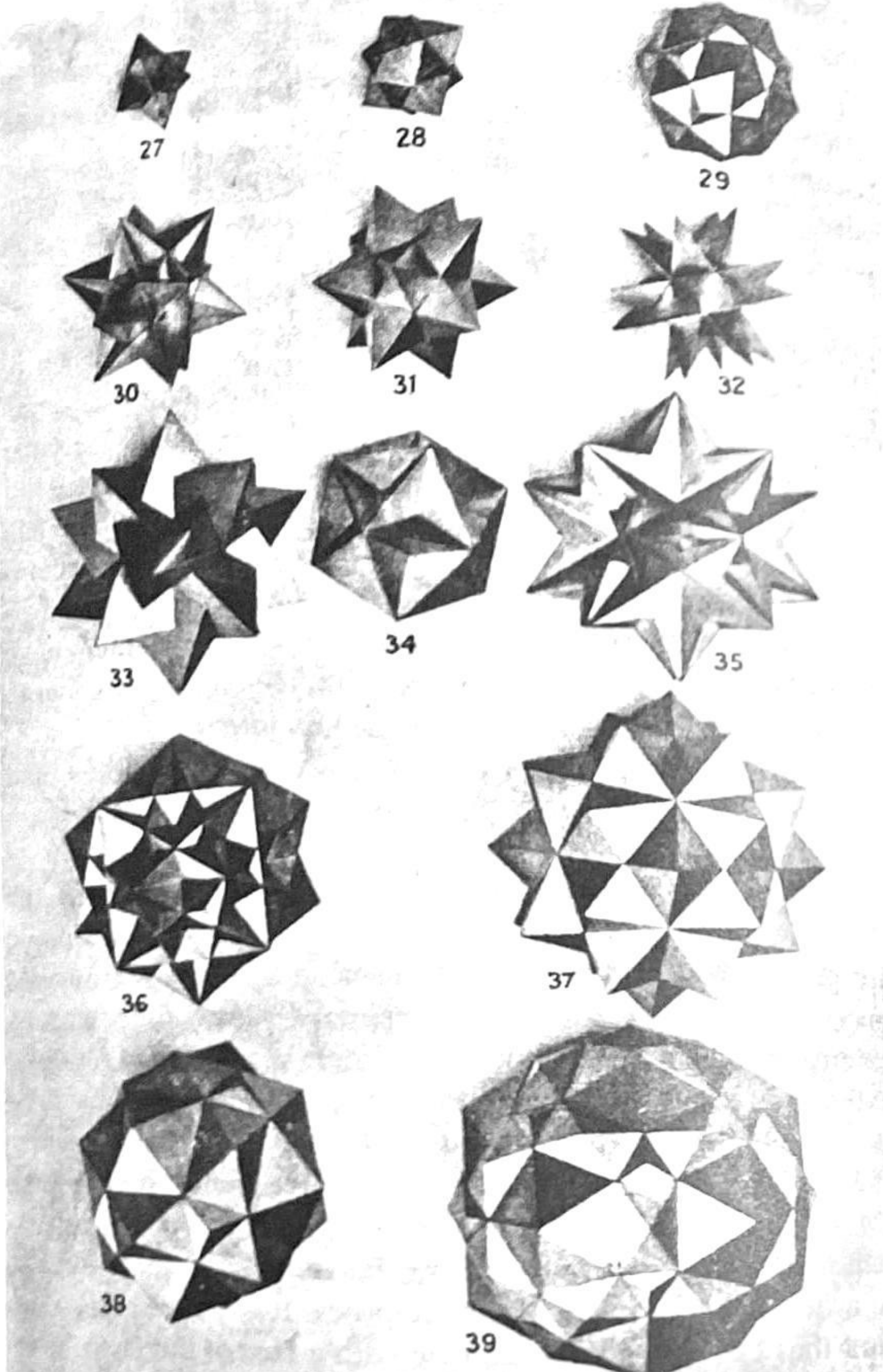
Opera Omnia, Frankfort, 1864, vol. v, p. 121.

of a polygon. Such polygons are the V faces of another "reciprocal" polyhedron, which has F vertices. The reciprocal of $\{p, q\}$ is $\{q, p\}$. Thus the cube and the octahedron are reciprocal, likewise the dodecahedron and the icosahedron. The tetrahedron is self-reciprocal or, rather, reciprocal to another tetrahedron. The diagonals of the faces of a cube are the edges of two reciprocal tetrahedra. (See figure 27 on plate II, facing this page.) The term *reciprocal* arises from the existence of a reciprocating sphere, with respect to which the vertices of $\{q, p\}$ are the poles of the face-planes of $\{p, q\}$, and vice versa. The ratio of circum-radius R to in-radius r is exactly the same for the cube as for the octahedron, and for the dodecahedron as for the icosahedron. In fact, if the reciprocating sphere has radius ρ, the reciprocal of a given polyhedron has circum-radius ρ^2/r and in-radius ρ^2/R. Thus the relative size of two reciprocal polyhedra may be adjusted so as to make them have the same circum-sphere and the same in-sphere. (In general, their corresponding edges will no longer intersect.)

If two reciprocal regular solids of the same in-radius (and therefore the same circum-radius) stand side by side on a horizontal plane (such as a table top), the distribution of vertices in horizontal planes is the same for both – i.e. the planes are the same, and the numbers of vertices in each plane are proportional. This fact was noticed by Pappus,* but has only recently been adequately explained, although its various extensions indicated that it was no mere accident. One of these extensions is to the Kepler-Poinsot polyhedra, which will be described later. Another is to tessellations of a plane. Consider the tessellation $\{6, 3\}$ (i.e. hexagons, three round each vertex). By picking out alternate vertices of each hexagon in a consistent manner, we derive the triangular tessellation $\{3, 6\}$ (which, in a different position, is the reciprocal tessellation). We then find that every circle concentric with a face of the $\{6, 3\}$ contains twice as many vertices of the $\{6, 3\}$ as of the $\{3, 6\}$. (This, however, is obvious, since the omitted vertices of the $\{6, 3\}$ belong to another $\{3, 6\}$, congruent to the first.)

*T. Heath, *A History of Greek Mathematics*, vol. II, pp. 368–369.

Plate II



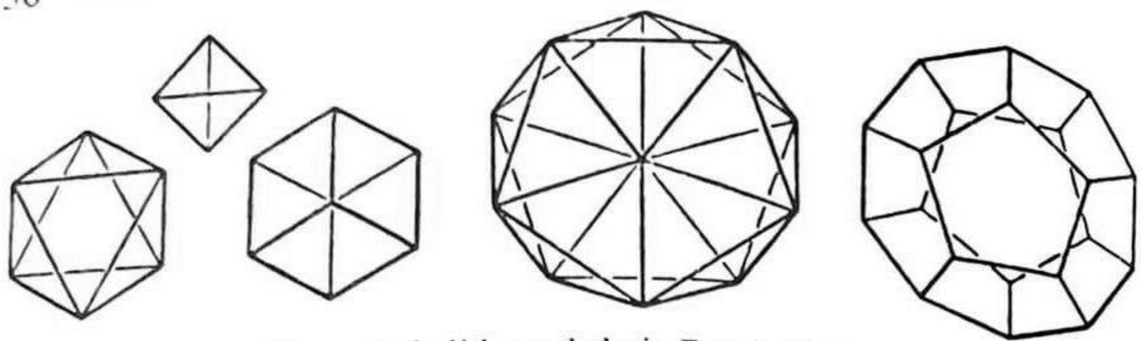
The fact that the vertices of a hexagonal tessellation belong also to two triangular tessellations is analogous to our observation (p. 134) that the vertices of a cube belong also to two regular tetrahedra. These two tetrahedra may be said to form a *compound* – Kepler's *stella octangula*; their eight faces lie in the facial planes of an octahedron. There is also a compound of *five tetrahedra* having the vertices of a dodecahedron and the facial planes of an icosahedron; this occurs in two enantiomorphous varieties. By putting the two varieties together, so as to have the same twenty vertices, we obtain a compound of *ten tetrahedra*, oppositely situated pairs of which can be replaced by *five cubes* (having the twenty vertices of the dodecahedron, each taken twice). It is quite easy to visualize one such cube in a given dodecahedron (as in the second drawing on page 133); the whole set of five makes a very pretty model. Finally, by reciprocating the five cubes we obtain a compound of *five octahedra* having the facial planes of an icosahedron, each taken twice. This icosahedron is inscribed in each one of the octahedra as in the first drawing on page 133. (See also plate II, figures 27, 33, 35, 36, 37.)

Among the edges of a regular polyhedron, we easily pick out a skew polygon or zigzag, in which the first and second edges are sides of one face, the second and third are sides of another face, and so on. This zigzag is known as a *Petrie polygon*, and has many applications. Each finite polyhedron can be orthogonally projected on to a plane in such a way that one Petrie polygon becomes a regular polygon with the rest of the projection inside it. It can be shown in various simple ways that the Petrie polygon of $\{p, q\}$ has h sides, where

$$\cos^2(\pi/h) = \cos^2(\pi/p) + \cos^2(\pi/q).$$

The h sides of the Petrie polygon of $\{p, q\}$ are crossed by h edges of the reciprocal polyhedron $\{q, p\}$; these form a Petrie polygon for $\{q, p\}$.

The regular polyhedra are symmetrical in many different ways. There is an *axis* of symmetry through the centre of every face, through the mid-point of every edge, and through



The Platonic Solids and their Petrie Polygons

every vertex: $E+1$ axes altogether. There are also $3h/2$ planes of symmetry.

THE ARCHIMEDEAN SOLIDS

A polyhedron is said to be *uniform* if it has regular faces and admits symmetries which will transform a given vertex into every other vertex in turn. The Platonic polyhedra are uniform; so are the right regular prisms and antiprisms, of suitable height – namely, when their lateral faces are squares and equilateral triangles, respectively. Such a polyhedron may be denoted by a symbol giving the numbers of sides of the faces around one vertex (in their proper cyclic order); thus the n-gonal prism and antiprism are $4^2.n$ and $3^3.n$. It is quite easy to prove* that, apart from these, there are just thirteen (finite, convex) uniform polyhedra:

$$3.6^2,\quad 4.6^2,\quad 3.8^2,\quad 5.6^2,\quad 3.10^2,\quad 4.6.8,\quad 4.6.10,$$
$$(3.4)^2,\quad (3.5)^2,\quad 3.4^3,\quad 3.4.5.4,\quad 3^4.4,\quad 3^4.5.$$

These are the *Archimedean solids.*

Let σ denote the sum of the face-angles at a vertex. (This must be less than 2π in order to make a solid angle.) Then the number of vertices is given by the formula† $(2\pi-\sigma)V=4\pi$. For instance, $3^4.5$ has 60 vertices, since $\sigma=(\frac{4}{3}+\frac{3}{5})\pi$.

If we regard the *stella octangula* as consisting of two interpenetrating solid tetrahedra, we may say that their common part is an octahedron. Also, as we have already observed,

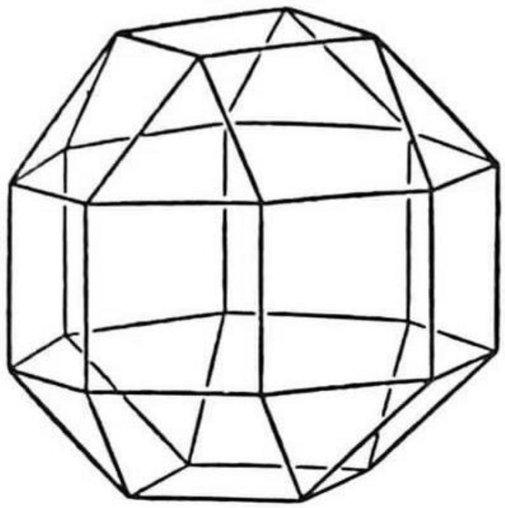
*See, for instance, T.R.S. Walsh, *Geometriae Dedicata*, 1972, vol. 1, pp. 117–123.

†E. Steinitz and H. Rademacher, *Vorlesungen über die Theorie der Polyeder*. Berlin. 1934, p. 11.

their edges are diagonals of the faces of a cube. Analogously, the common part of a cube and an octahedron, in the properly reciprocal position (with corresponding edges perpendicularly bisecting each other), is the *cuboctahedron* $(3.4)^2$. Each pair of corresponding edges (of the cube and octahedron) are the diagonals of a rhomb, and the twelve such rhombs are the faces of a "semi-regular" polyhedron known as the *rhombic dodecahedron.* (The latter is not uniform, but "isohedral." See the first drawing on page 151.) After suitable magnification, the edges of the cuboctahedron intersect those of the rhombic dodecahedron (at right angles); in fact, these two polyhedra are reciprocal, just as the octahedron and cube are reciprocal. The icosahedron and dodecahedron lead similarly to the *icosidodecahedron* $(3.5)^2$, and to its reciprocal, the *triacontahedron.* (See plate II, figures 28, 29, and plate I, figures 12, 10, 20, 18. Compare the tessellations drawn on page 106.) The compound of five cubes has the 30 facial planes of a triacontahedron. Reciprocally, the compound of five octahedra has the 30 vertices of an icosidodecahedron.

The faces of the icosidodecahedron consist of 20 triangles and 12 pentagons (corresponding to the faces of the two parent regulars). Its 60 edges are perpendicularly bisected by those of the reciprocal triacontahedron (although the latter edges are not bisected by the former: see plate II, figure 39). The 60 points where these pairs of edges cross one another are the vertices of a polyhedron whose faces consist of 20 triangles, 12 pentagons, and 30 rectangles. By slightly displacing these points (towards the mid-points of the edges of the triacontahedron), the rectangles can be distorted into squares, and we have another Archimedean solid, the *rhombicosidodecahedron,* 3.4.5.4. (Plate I, figure 23; compare the tessellation 3.4.6.4.) An analogous construction leads to the *rhombicuboctahedron** 3.4^3, whose faces consist of 8 triangles and 6+12 squares. (See plate II, figure 38, and plate I, figure 13.) In attempting

*I.e., "rhombi-cub-octahedron"; but the other is "rhomb-icosi-dodecahedron."



Pseudo-rhombicuboctahedron

to make a model of this polyhedron, J.C.P. Miller* accidentally discovered a "pseudo-rhombicuboctahedron," bounded likewise by 8 triangles and 18 squares, and isogonal in the loose or "local" sense (each vertex being surrounded by one triangle and three squares), but not in the strict sense (which implies that the appearance of the solid as a whole must remain the same when viewed from the direction of each vertex in turn).

On cutting off the corners of a cube, by planes parallel to the faces of the reciprocal octahedron, we leave small triangles, and reduce the square faces to octagons. For suitable positions of the cutting planes these octagons will be regular, and we have another Archimedean solid, the *truncated cube*, 3.8^2. (Cf. the tessellations 4.8^2 and 3.12^2.) Each of the five Platonic solids has its truncated variety;† so have the cuboctahedron and the icosidodecahedron, but in these last cases (4.6.8 and 4.6.10) a distortion is again required, to convert rectangles into squares.‡ (Cf. the tessellation 4.6.12.)

All the Archimedean solids so far discussed are reflexible (by reflection in the plane that perpendicularly bisects any edge). The remaining two, however, are not reflexible: the

**Philosophical Transactions of the Royal Society*, 1930, series A, vol. CCXXIX, p. 336.

†The "truncated $\{p, q\}$" is $q.(2p)^2$. See plate I, figures 11, 15, 16, 25, 22.

‡On account of this distortion, the truncated cuboctahedron (4.6.8) is sometimes called the "great rhombicuboctahedron," and then 3.4^3 is called the *small* rhombicuboctahedron; similarly for the truncated icosidodecahedron and rhombicosidodecahedron.

snub cube $3^4.4$, and the *snub dodecahedron* $3^4.5$ (plate I, figures 17 and 21). Let us draw one diagonal in each of the 30 squares of the rhombicosidodecahedron, choosing between the two possible diagonals in such a way that just one of these new lines shall pass through each of the 60 vertices. (The choice in the first squares determines that in all the rest.) Each square has now been divided into two right-angled isosceles triangles; by distorting these into equilateral triangles we obtain the snub dodecahedron.* The snub cube is similarly derivable from the rhombicuboctahedron, provided we remember to operate only on the 12 squares that correspond to the edges of the cube (and not on the 6 squares that correspond to its faces). The tessellation $3^4.6$ may be regarded as a "snub $\{6, 3\}$," and $3^2.4.3.4$ as a "snub $\{4, 4\}$." Moreover, the "snub tetrahedron" is the icosahedron $\{3, 5\}$, derived as above from the cuboctahedron (or "rhombi-tetra-tetrahedron").

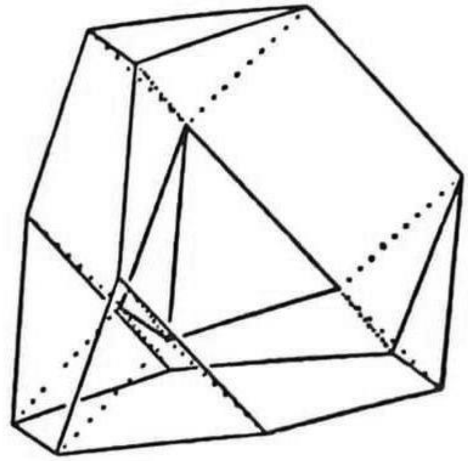
The snub cube and the snub dodecahedron both occur in two enantiomorphous varieties. Their metrical properties involve the solution of cubic equations, whereas those of the reflexible Archimedeans (and of the regulars) involve nothing worse than square roots; in other words, the reflexibles are capable of Euclidean construction, but the two proper snubs are not.

MRS. STOTT'S CONSTRUCTION

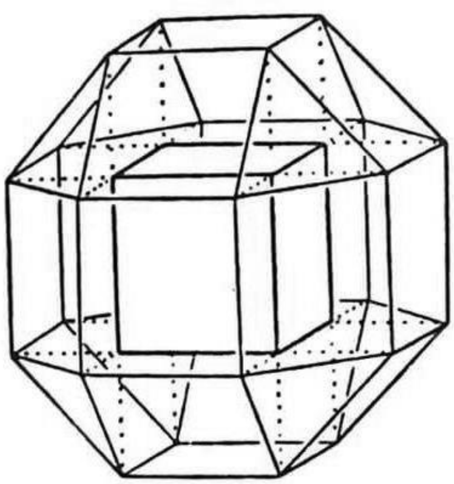
The above description of the Archimedean solids is essentially Kepler's. A far more elegant construction for the reflexible figures has been devised by Alicia Boole Stott.† Her method is free from any employment of distortion, and the final edge-length is the same as that of the regular solid from which we start. In the process called *expansion*, certain sets of elements (viz., edges or faces) are moved directly away from the centre, retaining their size and orientation, until the consequent in-

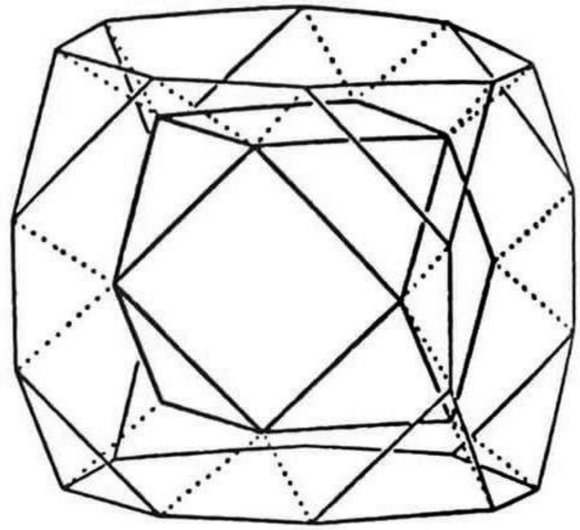
*This name is unfortunate, since the figure is related to the icosahedron just as closely as to the dodecahedron. "Snub icosidodecahedron" would be far better.

†*Verhandelingen der Koninklijke Akademie van Wetenschappen*, Amsterdam, 1910, vol. XI, no. 1.



(1) Tetrahedron and Truncated Tetrahedron





(2) Cube and Rhombicuboctahedron (3) Truncated Cube and Cuboctahedron

terstices can be filled with new regular faces. The reverse process is called *contraction*. By expanding any regular solid according to its edges, we derive the "truncated" variety. By expanding the cube (or the octahedron) according to its faces, we derive the rhombicuboctahedron, 3.4^3. By expanding this according to its 12 squares which correspond to the edges of the cube, or by expanding the truncated cube according to its octagons, we derive the truncated cuboctahedron, 4.6.8. By contracting the truncated cube according to its triangles, we derive the cuboctahedron. And so on. Mrs. Stott has represented these processes by a compact symbolism, and extended them to spaces of more than three dimensions, where they are extraordinarily fruitful.

EQUILATERAL ZONOHEDRA

The solids that I am about to describe were first investigated by E.S. Fedorov.* Their interest has been enhanced by P.S. Donchian's observation that they may be regarded as three-dimensional projections of n-dimensional *hyper-cubes* (or *measure-polytopes*, or *regular orthotopes*†). Their edges are all equal, and their faces are generally rhombs, but sometimes higher "parallel-sided $2m$-gons," i.e., equilateral $2m$-gons which are symmetrical by a half-turn. The subject begins with the following theorem on polygonal dissection.

Every parallel-sided $2m$-gon (and, in particular, every regular $2m$-gon) can be dissected‡ into $\frac{1}{2}m(m-1)$ rhombs of the same length of side. This is easily proved by induction, since every parallel-sided $2(m+1)$-gon can be derived from a parallel-sided $2m$-gon by adding a "ribbon" of m rhombs. In fact, the pairs of parallel sides of such a $2m$-gon can take any m different directions, and there is a component rhomb for every pair of these directions; hence the number $\frac{1}{2}m(m-1)$. For two perpendicular directions, the rhomb is a square.

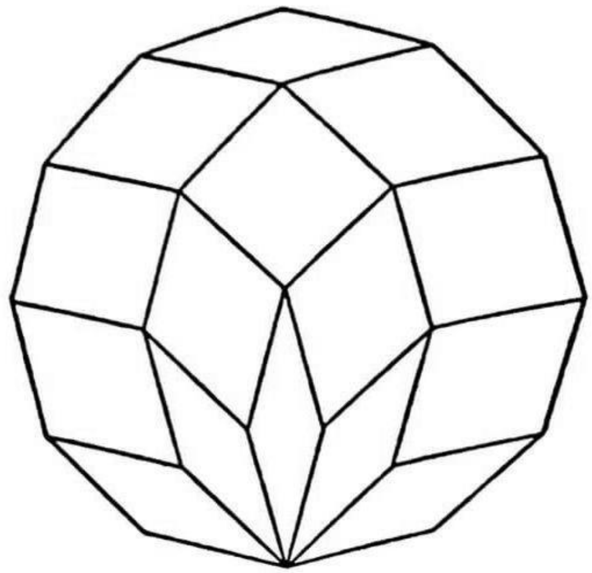
Consider now any sheaf of n lines through one point of space,§ and suppose first that no three of the lines are coplanar. Then there is a polyhedron whose faces consist of $n(n-1)$ rhombs, and whose edges, in sets of $2(n-1)$, are parallel to the n given lines. In fact, for every pair of the n lines, there is a pair of opposite faces whose sides lie in those directions. To construct this *equilateral zonohedron*, imagine a plane through any one of the n lines, gradually rotating through a complete turn. Each time that this plane passes through one

Zeitschrift für Krystallographie und Mineralogie, 1893, vol. XXI, p. 689; *Nachala Ucheniya o Figurakh*, Leningrad, 1953. See also Coxeter, *Twelve Geometric Essays*, Carbondale, Illinois, 1968, chap. 4.

†L. Schläfli, *Quarterly Journal of Mathematics*, 1860, vol. III, p. 66: "(4, 3, 3, . . ., 3)"; C.H. Hinton, *The Fourth Dimension*, London, 1906; P.H. Schoute, *Mehrdimensionale Geometrie*, Leipzig, 1905, vol. II, pp. 243–246; D.M.Y. Sommerville, *An Introduction to the Geometry of* n *Dimensions*, London, 1929, pp. 49, 171, 182, 190.

‡In how many ways?

§This construction is due to P.S. Donchian.



Fifteen Rhombs in a Dodecagon

of the other $n-1$ lines, take a rhomb whose edges are parallel to the two lines, and juxtapose it to the rhomb previously found (without changing its orientation). This process leads eventually to a closed ribbon of $2(n-1)$ rhombs. By fixing our attention on another of the n lines, we obtain another such ribbon, having two parallel faces in common with the first. When a sufficient number of these ribbons (or *zones*) have been added, the polyhedron is complete.

If m of the n lines are coplanar, we have a pair of opposite parallel-sided $2m$-gons, to replace $\frac{1}{2}m(m-1)$ pairs of opposite rhombs. If these m lines are symmetrically disposed, the $2m$-gons will be regular.

In this manner, three perpendicular lines lead to a cube, and three lines of general direction to a parallelepiped with rhombic faces. (This is called a *rhombohedron* only if the six faces are congruent.) More generally, m coplanar lines and one other line lead to a parallel-sided $2m$-gonal prism (a *right* prism if the "other" line is perpendicular to the plane of the first m).

The four "diameters" of the cube (joining pairs of opposite vertices) lead to the rhombic dodecahedron, the six diameters of the icosahedron lead to the triacontahedron, and the ten diameters of the (pentagonal) dodecahedron lead to an *ennea-*

*contahedron** whose faces are 30 rhombs of one kind and 60 of another. The six diameters of the cuboctahedron lead to the truncated octahedron, whose faces are 6 squares and 8 hexagons (the equivalent of 8×3 rhombs), and the fifteen diameters of the icosidodecahedron lead to the truncated icosidodecahedron, whose faces are 30 squares, 20 hexagons ($= 20 \times 3$ rhombs), and 12 decagons ($= 12 \times 10$ rhombs). As a final example, the nine diameters of the octahedron and cuboctahedron (taken together in corresponding positions)† lead to the truncated cuboctahedron, whose faces are 12 squares, 8 hexagons ($=24$ rhombs), and 6 octagons ($=36$ rhombs). (See plate I, figures 10, 18, 24, 16, 26, 19.)

As in these examples, so in general, the solid has the same type of symmetry as the given sheaf of lines. A rhombic $n(n-1)$-hedron having a centre of symmetry and an n-gonal axis occurs for every value of n, being given by a sheaf of n lines symmetrically disposed around a cone.‡ The faces are all alike when n is 3; they can be all alike when n is 4 or 5, if the lines are suitably chosen, viz., if the angle between alternate lines is supplementary to the angle between consecutive lines. Then $n=4$ gives the rhombic dodecahedron; $n=5$ gives a *rhombic icosahedron*§ (plate I, figure 14) which can be derived from the triacontahedron by removing any one of the zones and bringing together the two pieces into which the remainder of the surface is thereby divided. By removing a suitable zone from the rhombic icosahedron we obtain Bilinski's new rhombic dodecahedron with faces all alike but different from those of the classical rhombic dodecahedron.

Fedorov's general zonohedron can be derived from the equilateral zonohedron by lengthening or shortening all the edges that lie in each particular direction. Thus rhombic

*This somewhat resembles a figure described by A. Sharp, *Geometry Improv'd*, p. 87.

†I.e., perpendiculars to the nine planes of symmetry of the cube (or of the octahedron).

‡B.L. Chilton and H.S.M. Coxeter, *American Mathematical Monthly*, 1963, vol. LXX, pp. 946–951.

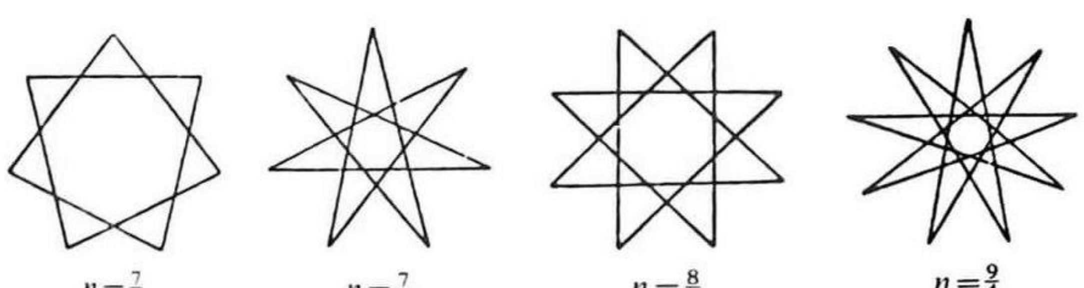
§Stanko Bilinski, *Glasnik*, 1960, vol. XV, pp. 252–262.

faces become parallelograms, and "parallel-sided $2m$-gons" cease to be equilateral. When each higher face is replaced by its proper number of parallelograms, we have $F=n(n-1)$, $E=2F$, and $V=F+2$. In fact, *every convex polyhedron bounded solely by parallelograms is a zonohedron.**

One final remark on this subject: there is a three-dimensional analogue for the theorem that a parallel-sided $2m$-gon can be dissected into $\frac{1}{2}m(m-1)$ parallelograms. The zonohedron can be dissected into $\frac{1}{6}n(n-1)(n-2)$ parallelepipeds (viz., one for every three of the n directions).

THE KEPLER-POINSOT POLYHEDRA

By extending the sides of a regular pentagon till they meet again, we derive the star-pentagon or pentacle or *pentagram*, which has long been used as a mystic symbol. We may regard the pentagram $\{\frac{5}{2}\}$ as a generalized polygon, having five sides which enclose the centre twice. Each side subtends an angle $\frac{4}{5}\pi$ at the centre, whereas each side of an ordinary n-gon subtends an angle $2\pi/n$. Thus the pentagram behaves as if it were an n-gon with $n=\frac{5}{2}$. Analogously, any rational number n (>2) leads to a polygon $\{n\}$, the numerator giving the number of sides, and the denominator the "density" (or "species").



$n=\frac{7}{2}$ $n=\frac{7}{3}$ $n=\frac{8}{3}$ $n=\frac{9}{4}$

This process of "stellation" may also be applied in space. The stellated faces of the regular dodecahedron meet by fives at twelve new vertices, forming the *small stellated dodecahedron* $\{\frac{5}{2}, 5\}$. These new vertices belong also to an icosahedron. By

*Coxeter, *Regular Polytopes*, New York, 1973, p. 27.

inserting the edges of this icosahedron, but keeping the original twelve facial planes, we obtain a polyhedron whose faces are twelve ordinary pentagons, while the section near a vertex is a pentagram; this is the *great dodecahedron*,* $\{5, \frac{5}{2}\}$. It is reciprocal to $\{\frac{5}{2}, 5\}$, as its symbol implies. By stellating the faces of $\{5, \frac{5}{2}\}$, we derive the *great stellated dodecahedron*, $\{\frac{5}{2}, 3\}$, which has the twenty vertices of an ordinary dodecahedron. Its reciprocal, the *great icosahedron*† $\{3, \frac{5}{2}\}$, has twenty triangular faces, and its vertices are those of an ordinary icosahedron. (See plate II, figures 31, 34, 32, 30.)

Thus we increase the number of finite regular polyhedra from five to nine. One way to see that these exhaust all the possibilities‡ is by observing that the "Petrie polygon" of $\{p, q\}$ is still characterized by the number h, where

$$\cos^2(\pi/h) = \cos^2(\pi/p) + \cos^2(\pi/q),$$

even when p and q are not integers. Writing this equation in the symmetrical form

$$\cos^2(\pi/p) + \cos^2(\pi/q) + \cos^2(\pi/k) = 1$$

(where $1/k = \frac{1}{2} - 1/h$), we find its rational solutions to be the three permutations of 3, 3, 4, and the six permutations of 3, 5, $\frac{5}{2}$, making nine in all, as required.

$\{p, q\}$	V	E	F	D	Name	Discoverer
$\{\frac{5}{2}, 5\}$	12	30	12	3	Small stellated dodecahedron	Kepler (1619)
$\{\frac{5}{2}, 3\}$	20	30	12	7	Great stellated dodecahedron	"
$\{5, \frac{5}{2}\}$	12	30	12	3	Great dodecahedron	Poinsot (1809)
$\{3, \frac{5}{2}\}$	12	30	20	7	Great icosahedron	"

*The *Encyclopaedia Britannica* (XIVth edition, article "Solids") unhappily calls this the "small stellated dodecahedron," and vice versa. (Cf. the XIth edition, article "Polyhedron.")

†Good drawings of these figures are given by Lucas (in his *Récréations mathématiques*), vol. II, pp. 206–208, 224.

‡This was first proved (another way) by Cauchy, *Journal de l'École Polytechnique*, 1813, vol. IX, pp. 68–86.

The polyhedra $\{\frac{5}{2}, 5\}$ and $\{5, \frac{5}{2}\}$ fail to satisfy Euler's Formula $V-E+F=2$, which holds for all ordinary polyhedra. The reason for this failure (which apparently induced Schläfli* to deny the existence of these two figures) will appear in chapter VIII. However, all the nine finite regular polyhedra satisfy the following extended theorem, due to Cayley:

$$d_V V - E + d_F F = 2D,$$

where d_F is the "density" of a face (viz., 1 for an ordinary polygon, 2 for a pentagram), d_V is the density of a vertex (or rather, of the section near a vertex), and D is the density of the whole polyhedron (i.e. the number of times the faces enclose the centre).

"Archimedean" star polyhedra have been investigated,† but are beyond the scope of this book.

THE 59 ICOSAHEDRA

Imagine a large block of wood with a small tetrahedron or cube (somehow) drawn in the middle. If we make saw-cuts along all the facial planes of the small solid, and throw away all the pieces that extend to the surface of the block, nothing remains but the small solid itself. But if, instead of a tetrahedron or cube, we start with an octahedron, we shall be left with nine pieces: the octahedron itself, and a tetrahedron on each face, converting it into a *stella octangula* which has the appearance of two interpenetrating tetrahedra (the regular compound mentioned above). Similarly, a dodecahedron leads to $1+12+30+20$ pieces: the dodecahedron itself, twelve pentagonal pyramids which convert this into the small stellated dodecahedron, thirty wedge-shaped tetrahedra which convert the latter into the great dodecahedron, and twenty triangular double-pyramids which convert this last into the great stellated dodecahedron.

**Quarterly Journal of Mathematics*, 1860, vol. III, pp. 66, 67. He defined "$(\frac{5}{2}, 3)$, $(3, \frac{5}{2})$," but not "$(\frac{5}{2}, 5)$, $(5, \frac{5}{2})$."

†Coxeter, Longuet-Higgins, and Miller, *Philosophical Transactions of the Royal Society*, 1954, series A, vol. CCXLVI, pp. 401–450.

Finally, the icosahedron* leads to 1+20+30+60+20+60 +120+12+30+60+60 pieces, which can be put together to form 32 different reflexible solids, all having the full icosahedral symmetry, and 27 pairs of enantiomorphous solids, having only the symmetry of rotation. The former set of solids includes the original icosahedron, the compound of five octahedra (made of the first 1+20+30 pieces), the compound of ten tetrahedra (made of the first 1+20+30+60+20+60+120 pieces), and the great icosahedron (made of all save the last 60 pieces). The latter set includes the compound of five tetrahedra, and a number of more complicated figures having the same attractively "twisted" appearance.†

SOLID TESSELLATIONS

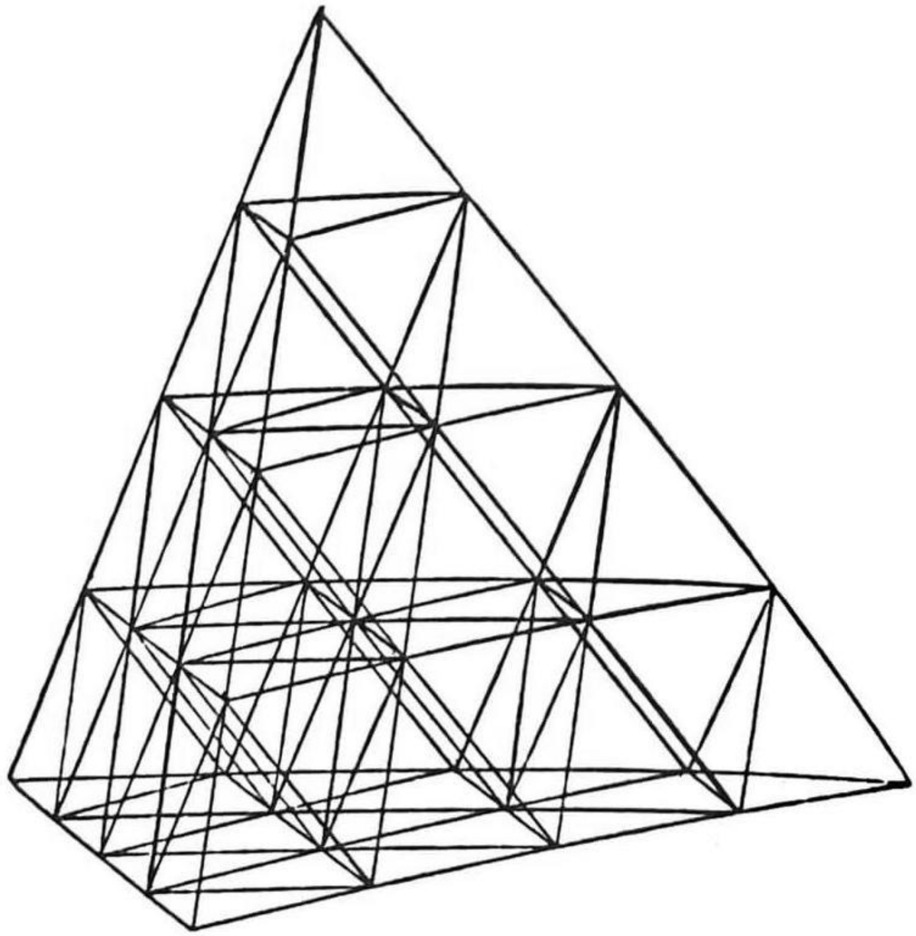
Just as there are many symmetrical ways of filling a plane with regular polygons, so there are many symmetrical ways of filling space with regular and Archimedean solids. For the sake of brevity, let us limit our discussion to those ways in which all the edges (as well as all the vertices) are surrounded alike. Of such "solid tessellations" there are just five,‡ an edge being surrounded by (i) four cubes, or (ii) two tetrahedra and two octahedra, arranged alternately, or (iii) a tetrahedron and three truncated tetrahedra, or (iv) three truncated octahedra, or (v) an octahedron and two cuboctahedra. Let us denote these by the symbols $[4^4]$, $[3^4]$, $[3^2.6^2]$, $[4.6^2]$, $[3^2.4]$, which indicate the polygons (interfaces) that meet at an edge.

The "regular" space-filling $[4^4]$ is familiar. It is "self-reciprocal" in the sense that the centres of all the cubes are the vertices of an identical space-filling. Its alternate vertices

*A.H. Wheeler, *Proceedings of the International Mathematical Congress, Toronto*, 1924, vol. I, pp. 701–708; M. Brückner, *Vielecke und Vielflache*, Leipzig, 1900 (plate VIII, nos. 2, 26; plate IX, nos. 3, 6, 11, 17, 20; plate X, no. 3; plate XI, nos. 14, 24).

†For J.F. Petrie's exquisite drawings of all these figures, see "The 59 Icosahedra," *University of Toronto Studies* (Mathematical Series), no. 6, 1938.

‡A. Andreini, *Memorie della Società italiana delle Scienze*, 1905, series 2, vol. XIV, pp. 75–129, figs. 12, 15, 14, 18, 33.



The Solid Tessellation $[3^4]$

give the space-filling $[3^4]$, one tetrahedron being inscribed in each cube, and one octahedron surrounding each omitted vertex. This has a particularly high degree of regularity (although its solids are of two kinds, unlike those of $[4^4]$); for, not merely the vertices and edges, but also the triangular interfaces, are all surrounded alike; in fact, each triangle belongs to one solid of either kind. If we join the centres of adjacent solids, by lines perpendicular to the interfaces, and by planes perpendicular to the edges, we obtain the "reciprocal" space-filling, say $[3^4]'$; this consists of rhombic dodecahedra, of which four surround some vertices (originally centres of tetrahedra), while six surround others (originally centres of octahedra).

The space-filling $[3^2.6^2]$ can be derived from $[3^4]$ by making each of a certain set of tetrahedra of the latter adhere to its four adjacent octahedra and to six other tetrahedra which connect these in pairs, so as to form a truncated tetrahedron.* Thus $[3^2.6^2]$ has half the vertices of $[3^4]$, which in turn has half the vertices of $[4^4]$.

The space-filling of truncated octahedra, $[4.6^2]$, is reciprocal to a space-filling of "isosceles" tetrahedra (or tetragonal bisphenoids) whose vertices belong to two reciprocal $[4^4]$'s (the "body-centred cubic lattice" of crystallography). The vertices of $[3^2.4]$ are the mid-points of the edges (*or* the centres of the squares) of $[4^4]$.

BALL-PILING OR CLOSE-PACKING

A large box can be filled with a number of small equal spheres arranged in horizontal layers, one on top of another, in various ways, of which I will describe three. It might be filled so that each sphere rests on the top of the sphere immediately below it in the next layer, touches each of four adjacent spheres in the same layer, and touches one sphere in the layer above it; thus each sphere is in contact with six others. Or we might slightly spread out the spheres of each layer, so as to be not quite in contact, and let each sphere rest on four in the layer below and help to support four in the layer above, the "spreading out" being adjusted so that the points of contact are at the vertices of a cube. We might also fill the box with spheres arranged so that each of them is in contact with four spheres in the next lower layer, with four in the same layer, and with four in the next higher layer. This last arrangement is known as *normal piling* or *spherical close-packing*; it gives the greatest number of spheres with which the box can be filled. (Although it is impossible for one sphere to touch more than twelve others of the same size, we shall see later that there are many different ways of packing equal spheres so that each touches exactly twelve others.)

These three arrangements may be described as follows. In the first, the centres of the spheres are at the vertices of the space-filling $[4^4]$, and the spheres themselves are inscribed in

*Analogously, any of the plane tessellations 6^3, $(3.6)^2$, $3^4.6$ may be derived from 3^6 by making certain sets of six triangles coalesce to form hexagons.

the cubes of the reciprocal $[4^4]$. In the second, the spheres are inscribed in the truncated octahedra of $[4.6^2]$ (touching the hexagons, but just missing the squares). In the third, the spheres are inscribed in the rhombic dodecahedra of $[3^4]'$, and their centres are at the vertices of $[3^4]$.

Now, the vertices of $[3^4]$ form triangular tessellations 3^6 in a series of parallel planes.

A *A* *A* *A*

B *B* *B*

C *C* *C* *C*

A *A* *A*

B *B* *B* *B*

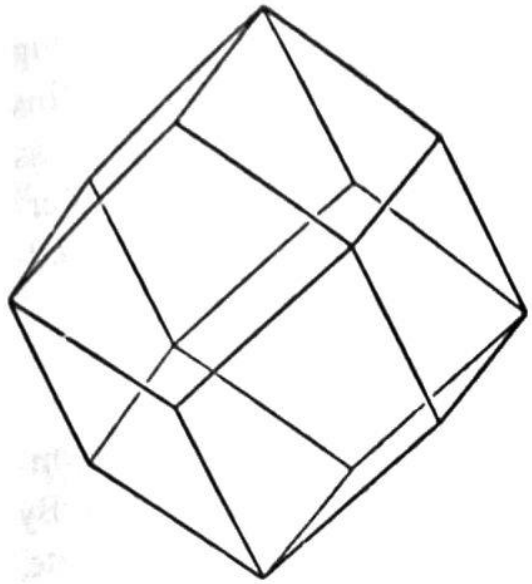
C *C* *C*

A *A* *A* *A*

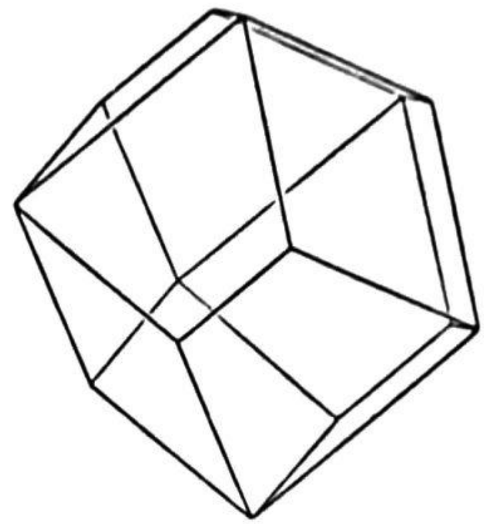
B *B* *B*

C *C* *C* *C*

Our figure shows a "plan" of this arrangement of points, projected orthogonally on one of the planes, which we take to be horizontal. The points *A* are projected from one plane, the points *B* from the next, *C* from the next, *A* again from the next, and so on, in cyclic order. Now imagine solid spheres centred at all these points. The points *A* give a layer of close-packed spheres, each touching six others. The points *B* represent another such layer, resting on the first; each sphere of either touches three of the other. The points *C* represent a third layer, resting on the second; but an equally "economical" piling of spheres is obtained if the centres of the third layer lie above the points *A* again, instead of lying above the points *C*. And so, at every stage, the new layer may or may not lie vertically above the last but one.



(1) Rhombic Dodecahedron



(2) Trapezo-rhombic Dodecahedron

The arrangement *A B C A B C* . . . represents spherical close-packing; on the other hand, the arrangement *A B A B A B* . . . is known as *hexagonal* close-packing. In both cases space is filled to the extent of 74 per cent. If a large number of equal balls of "plasticene" or modelling clay are rolled in chalk, packed in either fashion, and squeezed into a solid lump, those near the middle tend to form rhombic dodecahedra or trapezorhombic dodecahedra* respectively. If, instead of being carefully stacked, the balls are shaken into a random arrangement as dense as possible, and are then squeezed as before, the resulting shapes are irregular polyhedra of various kinds. The average number of faces† is not 12 but about 13·3. Equal spheres arranged in such a *random* piling have not been proved to occupy less space than the same spheres in normal piling; but it is clear that any small displacement will increase the total volume by enlarging the interstices.

If you stand on wet sand, near the sea-shore, it is very noticeable that the sand gets comparatively dry around your feet, whereas the footprints that you leave contain free water. The following explanation is due, I believe, to Osborne Reynolds. The grains of sand, rolled into approximately spherical shape

*Cf. Steinhaus, *Mathematical Snapshots*, New York, 1938, p. 88.
†J.D. Bernal, *Nature*, 1959, vol. CLXXXIII, pp. 141–147; Coxeter, *Introduction to Geometry*, pp. 410–412.

by the motion of the sea, have been deposited in something like random piling. The pressure of your feet disturbs this piling, increasing the interstices between the grains. Water is sucked in from around about, to fill up these enlarged interstices. When you remove your feet, the random piling is partially restored, and the water is left above.

REGULAR SPONGES

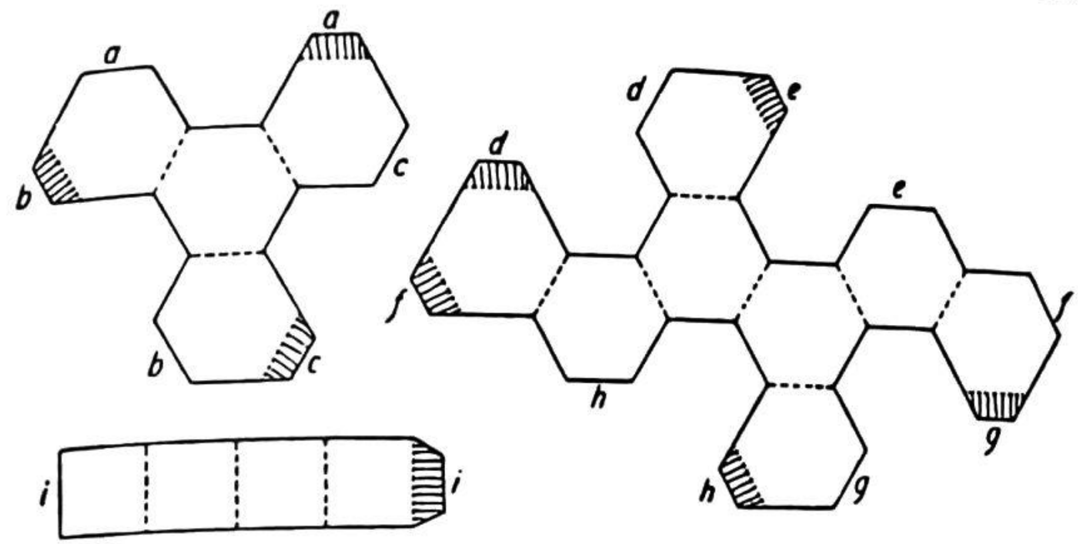
The definition of regularity on page 131 depends on two symmetries, which, in every case so far discussed, are *rotations*. By allowing the number of vertices, edges, and faces to be infinite, this definition includes the plane tessellations $\{3, 6\}$, $\{6, 3\}$, $\{4, 4\}$. It would be absurd to allow each face to have infinitely many sides, or to allow infinitely many faces to surround one vertex; therefore the special symmetries must be periodic. However, they need not be rotations; they may be rotatory-reflections. (A rotatory-reflection is the combination of a rotation and a reflection, which may always be chosen so that the axis of the rotation is perpendicular to the reflecting plane.) Such an operation interchanges the "inside" and "outside" of the polyhedron; consequently the inside and outside are identical, and the polyhedron (dividing space into two equal parts) must be infinite. The dihedral angles at the edges of a given face are alternately positive and negative, and the edges at a vertex lie alternately on the two sides of a certain plane. This allows the sum of the face-angles at a vertex to exceed 2π.

It can be proved that the polyhedra $\{p, q\}$ of this type are given by the integral solutions of the equation

$$2 \sin (\pi/p) \sin (\pi/q) = \cos (\pi/k),$$

namely $\{6, 6\}$ $(k=3)$, $\{6, 4\}$ and $\{4, 6\}$ $(k=4)$, $\{3, 6\}$ $(k=6)$, and $\{4, 4\}$ $(k=\infty)$. The three plane tessellations occur because a plane rotation may be regarded indifferently as a rotation in space or as a rotatory-reflection. The three new figures are "sponges" with k-gonal holes.*

*For photographs of models, see Coxeter, *Twelve Geometric Essays*, p. 77, where the three sponges are denoted by $\{6,6|3\}$, $\{6,4|4\}$, $\{4,6|4\}$.



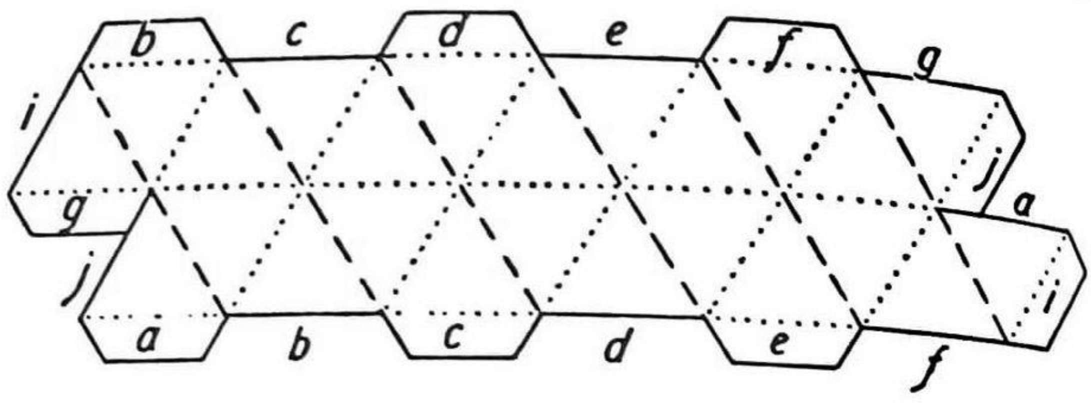
The faces of {6, 6} are the hexagons of the solid tessellation $[3^2.6^2]$; those of {6, 4} are the hexagons of $[4.6^2]$; and those of {4, 6} are half the squares of $[4^4]$. The remaining interfaces of the solid tessellations appear as holes. The last two sponges (discovered by J.F. Petrie in 1926) are *reciprocal*, in the sense that the vertices of each are the face centres of the other;* {6, 6} is self-reciprocal or, rather, reciprocal to another congruent {6, 6}.

To make a model of {6, 6}, cut out sets of four hexagons (of thin cardboard), stick each set together in the form of the hexagonal faces of a truncated tetrahedron (3.6^2), and then stick the sets together, hexagon on hexagon, taking care that no edge shall belong to more than two faces. (In the finished model, the faces are double, which makes for greater strength besides facilitating the construction.) Similarly, to make {6, 4}, use sets of eight hexagons, forming the hexagonal faces of truncated octahedra (4.6^2). Finally, to make {4, 6}, use rings of four squares. This last model, however, is not rigid; it can gradually collapse, the square holes becoming rhombic. (In fact, J.C.P. Miller once made an extensive model and mailed it, flat in an envelope).

*Plane tessellations can be reciprocal in this sense, but finite polyhedra cannot. The centres of the faces of an octahedron are the vertices of a cube, while the vertices of the octahedron are the centres of the faces of another (larger) cube.

ROTATING RINGS OF TETRAHEDRA

J.M. Andreas and R.M. Stalker have independently discovered a family of non-rigid finite polyhedra having $2n$ vertices, $6n$ edges (of which $2n$ coincide in pairs), and $4n$ triangular faces,



for $n=6$ or 8 or any greater integer. The faces are those of n tetrahedra, joined together in cyclic order at a certain pair of opposite edges of each, so as to form a kind of ring. When $n=6$, the range of mobility is quite small, but when $n=8$, the ring can turn round indefinitely, like a smoke-ring. When n is even, the figure tends to take up a symmetrical position; it is particularly pretty when $n=10$.* When n is odd, the entire lack of symmetry seems to make the motion still more fascinating. When $n \geqq 22$, the ring can occur in a knotted form.

A model of any such ring may be made from a single sheet of paper. For the case when $n=6$, copy the above diagram, cut it out, bend the paper along the inner lines, upwards or downwards according as these lines are broken or dotted, and stick the flaps in the manner indicated by the lettering. The ends have to be joined somewhat differently when n is a multiple of 4 (see figure XXXIV, page 216). When n is odd, either method of joining can be used at will.

Since there are two types of edge, such a polyhedron is not regular, and no symmetry is lost by making the triangles isosceles instead of equilateral. If the doubled edges are sufficiently

*One of the "Stephanoids" described by M. Brückner in his *Vielecke und Vielflache*, p. 216 (and plate VIII, no. 4) consists of a ring of ten irregular tetrahedra.

short compared to the others, the ring with $n=6$* can be made to turn completely, like the rings with $n \geqq 8$.

THE KALEIDOSCOPE†

The ordinary kaleidoscope consists essentially of two plane mirrors, inclined at $\pi/3$ or $\pi/4$, and an object (or set of objects) placed in the angle between them so as to be reflected in both. The result is that the object is seen 6 or 8 times (according to the angle), in an attractively symmetrical arrangement. By making a hinge to connect two (unframed) mirrors, the angle between them can be varied at will, and it is clear that an angle π/n gives $2n$ images (including the object itself). As a limiting case, we have two parallel mirrors and a theoretically infinite number of images (restricted in practice only by the brightness of the illumination and the quality of the mirrors). If the object is a point on the bisector of the angle between the mirrors, the images are the vertices of a regular $2n$-gon. If the object is a point on one of the mirrors, the images coincide in pairs at the vertices of a regular n-gon. The point may be represented in practice by a candle, or by a little ball of plastic clay or putty.

Regarding the two mirrors as being vertical, let us introduce a third vertical mirror in such a way that each pair of the three mirrors makes an angle of the form π/n. In other words, any horizontal section is to be a triangle of angles π/l, π/m, π/n, where l, m, n are integers. The solutions of the consequent equation

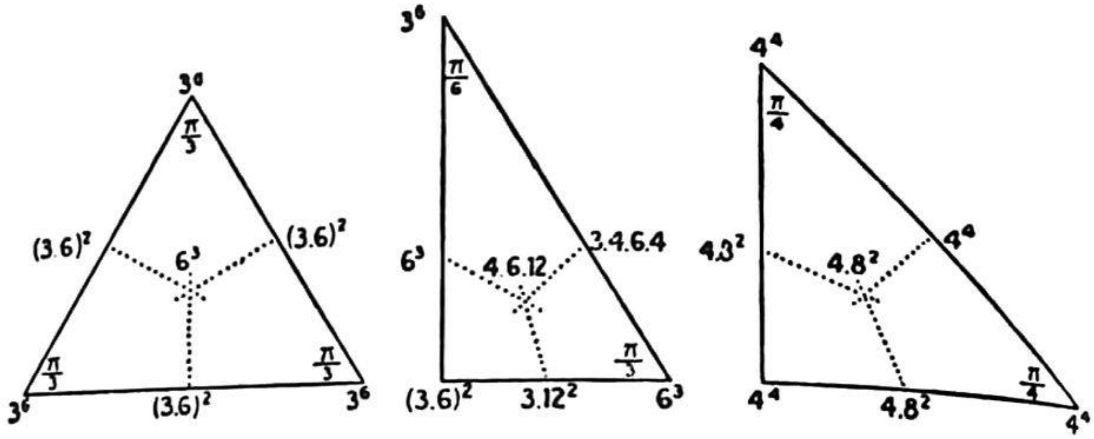
$$\frac{1}{l}+\frac{1}{m}+\frac{1}{n}=1$$

are 3, 3, 3; 2, 3, 6; 2, 4, 4. In each case the number of images is infinite. By varying the position of a point-object in the triangle, we obtain the vertices of certain isogonal tessella-

*Such a ring (of six tetragonal bisphenoids) has been on sale in the United States as a child's toy, with letters of the alphabet on its 24 faces. (Patent No. 1,997,022, issued in 1935.) See also M. Goldberg, *Journal of Mathematics and Physics*, 1947, vol. XXVI, pp. 10–21.

†E. Hess, *Neues Jahrbuch für Mineralogie, Geologie und Palaeontologie*, 1889, vol. I, pp. 54–65.

tions.* In particular, if the point is taken at a vertex of the triangle, or where an angle-bisector meets the opposite side, or at the in-centre (where all three angle-bisectors concur), then the tiles of the tessellation are regular polygons. In the notation of page 106, the results of putting the point in these various positions are as indicated in the following diagrams.



The network of triangles, which the mirrors appear to create, may be coloured alternately white and black. By taking a suitable point within every triangle of one colour (but ignoring the corresponding point within every triangle of the other colour), we obtain the vertices of 3^6 (again), $3^4.6$, and $3^2.4.3.4$, respectively. (The remaining uniform tessellation, $3^3.4^2$, is not derivable by any such method.) The above diagrams reveal many relationships between the various tessellations: that the vertices of 3^6 occur among the vertices of 6^3, that the vertices of 6^3 trisect the edges of (another) 3^6, that the vertices of one 4^4 bisect the edges of another, and so on.

If the third mirror is placed horizontally instead of vertically – i.e. if the two hinged mirrors stand upon it – the number of images is no longer infinite; in fact, it is $4n$, where π/n is the angle between the two vertical mirrors. For a point on one of the vertical mirrors, the images coincide in pairs at the vertices of an n-gonal prism. Two of the three dihedral angles

*Placing a lighted candle between three (unframed) mirrors, the reader will see an extraordinarily pretty effect. The University of Minnesota has made use of this idea in two short films: *Dihedral Kaleidoscopes* and *The Symmetries of the Cube*.

between pairs of the three mirrors are now right angles. A natural generalization is the case where these three angles are π/l, π/m, π/n.

Since, for any reflection in a plane mirror, object and image are equidistant from the plane, we easily see that all the images of a point in this generalized kaleidoscope lie on a sphere, whose centre is the point of intersection of the planes of the three mirrors. On the sphere, these planes cut out a spherical triangle, of angles π/l, π/m, π/n. The resulting image-planes divide the whole sphere into a network (or "map") of such triangles, each containing one image of any object placed within the first triangle. The number of images is therefore equal to the number of such triangles that will suffice to fill the whole spherical surface. Taking the radius as unity, the area of the whole sphere is 4π, while that of each triangle is $(\pi/l)+(\pi/m)+(\pi/n)-\pi$. Hence the required number is

$$4\bigg/\left(\frac{1}{l}+\frac{1}{m}+\frac{1}{n}-1\right).$$

Since this must be positive, the numbers l, m, n have to be chosen so as to satisfy

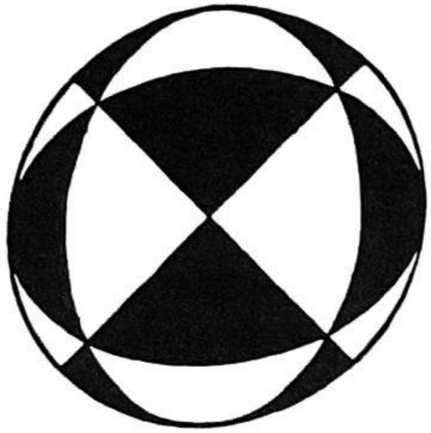
$$\frac{1}{l}+\frac{1}{m}+\frac{1}{n}>1.$$

This inequality has the solutions 2, 2, n; 2, 3, 3; 2, 3, 4; 2, 3, 5. The first case has already been mentioned; the rest are depicted on page 158 (by J.F. Petrie).

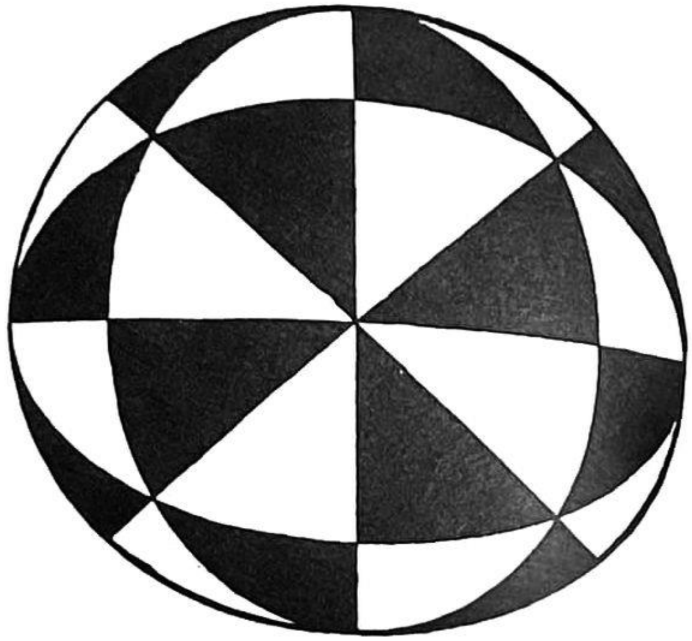
For a practical demonstration, the mirrors should in each case be cut as circular sectors (of the same fairly large radius), whose angles* are equal to the sides of a spherical triangle of angles π/l, π/m, π/n.

By varying the position of a point-object in the spherical triangle (or in the solid angle between the three mirrors) we

*In the three cases, these angles are respectively: 54°44′, 54°44′, 70°32′; 35°16′, 45°, 54°44′; 20°54′, 31°43′, 37°23′.



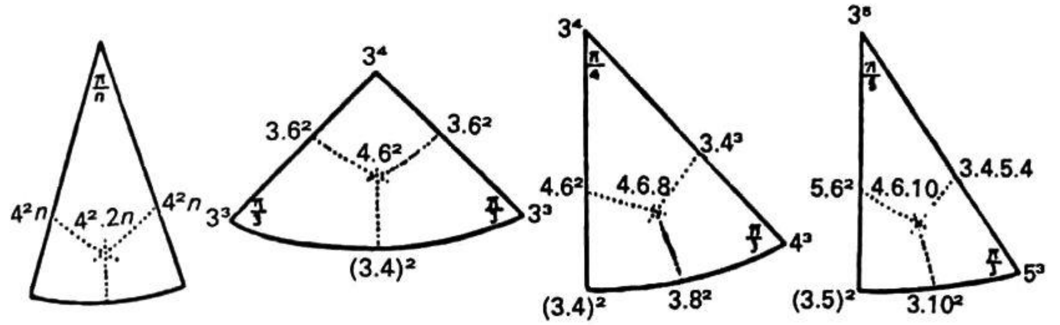
(1)



(2)

(3)

obtain the vertices of certain isogonal polyhedra. In particular, if the point is on one of the edges where two mirrors meet, or on one of the mirrors and equidistant from the other two, or at the centre of a sphere which touches all three, then the faces of the polyhedra are regular polygons. The manner in which the various uniform polyhedra arise* is indicated in the following diagrams, analogous to those given for tessellations on page 156.



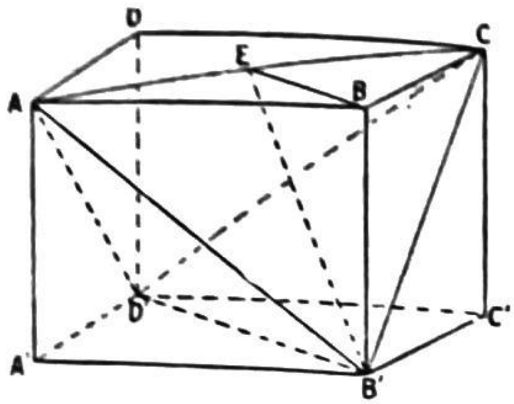
By taking a suitable point within each white (or black) triangle,† we obtain the vertices of $3^3.n$, 3^5, $3^4.4$, or $3^4.5$, respectively. It has already been remarked that the snub cube $3^4.4$ exists in two enantiomorphous forms; the vertices of one form lie in the white triangles, those of the other in the black. The same thing happens in the case of the snub dodecahedron, $3^4.5$.

By introducing a fourth mirror, we obtain solid tessellations. Tetrahedra of three different shapes can be formed by four planes inclined at angles that are submultiples of π. These three shapes can conveniently be cut out from a rectangular block of dimensions $1 \times \sqrt{2} \times \sqrt{2}$. Suppose $ABCD$ to be a horizontal square of side $\sqrt{2}$, at height 1 above an equal square $A'B'C'D'$. After cutting off the alternate corners A', B, C', D, by planes through sets of three other vertices, we are left with the tetragonal bisphenoid $AB'CD'$, which is one of the required shapes.

*See Möbius, *Gesammelte Werke*, 1861, vol. II, p. 656, figs. 47, 51, 54; W.A. Wythoff, *Proceedings of the Royal Academy of Sciences, Amsterdam*, 1918, vol. XX, pp. 966–970; G. de B. Robinson, *Journal of the London Mathematical Society*, 1931, vol. VI, pp. 70–75; H.S.M. Coxeter, *Proceedings of the London Mathematical Society*, 1935, series 2, vol. XXXVIII, pp. 327–339.

†Möbius, *loc. cit.*, figs. 46, 49, 53.

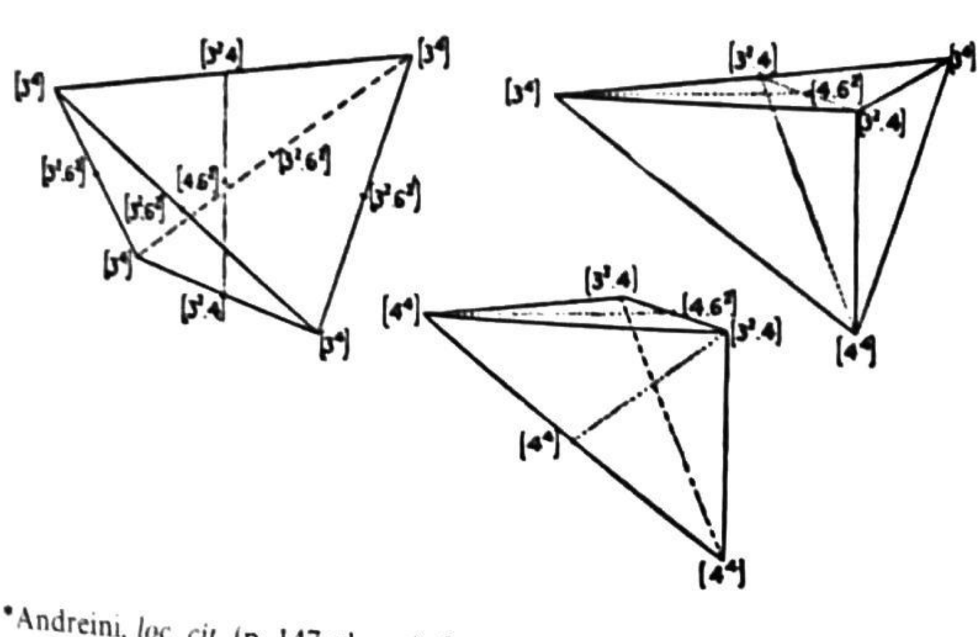
Another is any one of the corner pieces that were cut off, such as $ABCB'$. (Two such pieces can be fitted together to make a shape just like $AB'CD'$.) The third is obtained by cutting $ABCB'$ in half along its plane of symmetry, which is the plane $BB'E$,



where E is the mid-point of AC. One half is $AEBB'$. Note that the edges AE, EB, BB' are three equal lines in three perpendicular directions.

A point-object in such a tetrahedron will give rise to the vertices of a solid tessellation in various ways,* some of which are indicated in the following diagrams (which show $AB'CD'$, $ABCB'$, $AEBB'$, in the same orientation as before).

Five mirrors can be arranged in the form of certain triangular prisms; these lead to solid tessellations of prisms. Six can be arranged rectangularly in three pairs of parallels, as when we



*Andreini, *loc. cit.* (p. 147 above), figs. 17–24 bis.

have a mirror in the ceiling and floor as well as all four walls of an ordinary room; these give a solid tessellation of rectangular blocks. G. Pólya has proved* that any kaleidoscope is effectively equivalent to one having at most six mirrors.

* *Annals of Mathematics*, 1934, vol. xxxv, p. 594.

ADDENDUM

Note. Page 143. A *rhombohedron* is a parallelepiped bounded by six congruent rhombs. It has two opposite vertices at which the three face-angles are equal; it is said to be *acute* or *obtuse* according to the nature of these angles. A *golden* rhombohedron has faces whose diagonals are in the "golden" ratio $\tau : 1$ (see page 56). The Japanese architect Koji Miyazaki observed in 1977 that ten golden rhombohedra, five acute (A_6) and five obtuse (O_6), can be fitted together to form a rhombic icosahedron (F_{20}), and that two rhombic icosahedra, symmetrically placed with a common "obtuse" vertex, can be surrounded by further rhombohedra (thirty acute and thirty obtuse) to form a large rhombic icosahedron whose edges are twice as long; symbolically

$$30\ A_6 + 30\ O_6 + 2\ F_{20} = F_{20}\ 2.$$

Consequently the whole space can be filled with golden rhombohedra so as to form a honeycomb having a pentagonal axis of symmetry.

A different space-filling of A_6's and O_6's was discovered independently, at about the same time, by Robert Ammann in Massachusetts. He was seeking a 3-dimensional analogue for Roger Penrose's non-periodic tilings of the plane (see the *Mathematical Intelligencer*, 1979, vol. II, p. 36). Ammann's procedure was extended by the Japanese physicist Tohru Ogawa, whose idea is to iterate the construction

$$55\ A_6 + 34\ O_6 = A_6\ \tau^3, \qquad 34\ A_6 + 21\ O_6 = O_6\ \tau^3.$$

Since the volumes of A_6 and O_6 are in the ratio τ to 1, Ogawa's hierarchic rule illustrates the identities

$$55\tau + 34 = \tau^{10}, \qquad 34\tau + 21 = \tau^9$$

(which involve the same consecutive Fibonacci numbers 21, 34, 55 as the paradox on page 86, although now everything is exact). Since all the edges of this *quasilattice* are parallel to the six diameters joining pairs of opposite vertices of the regular icosahedron, icosahedral symmetry occurs in a statistical sense, agreeing with the apparent icosahedral symmetry of certain alloys of aluminium and manganese (see A.L. Mackay, *Nature*, 1986, vol. CCCXIX, pp. 102–104).