On the Four-Dimensional Angles of the Semiregular Polytopes of S4

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On the Four-dimensional Angles of the Semiregular Polytopes of S_4.

By P. H. Schoute.

§ 1. *Introduction.*

This paper may be regarded as a continuation of a previous one published in this Journal (Vol. XXXI, p. 303); it is concerned with polytopes of S_4 characterized by the property of admitting one kind of vertex and one length of edge, which polytopes will be called "semiregular." These polytopes, corresponding to the Archimedian semiregular polyhedra of ordinary space, have been deduced from the regular ones by very simple geometrical operations called "expansions" and "contractions" in a masterly memoir of Mr. A. Boole Stott;* they will be indicated here by the symbols introduced in that memoir.

It follows immediately from the property of admitting one kind of vertex and one length of edge that the portion of four-dimensional space occupied by a semiregular polytope at any vertex is the same for all the vertices of that polytope, We wish to determine this angle in parts of 16 right angles or 1440°; to that end we put on duty the space fillings or nets of semiregular polytopes. The nets derived from the regular cells C_8, C_{16}, C_{24} have been tabulated in the memoir quoted, those derived from the simplex C_5 have been added by ourselves.† As there are no nets consisting of polytopes, derived from the cells C_{120}, C_{600}, we discard the semiregular polytopes deduced from them.

There is a remarkable difference between the nets derived from C_8, C_{16}, C_{24}, on one hand, and those derived from C_5 on the other. With exception of the regular nets built up exclusively either of C_8 or of C_{16} or of C_{24}, any net of the first group admits, besides *principal* constituents derived from C_8, C_{16}, C_{24},

* "Geometrical Deduction of Semiregular from Regular Polytopes and Space Fillings," *Verhandelingen* of Amsterdam, Vol. XI (1910), No. 1.

† "Analytical Treatment of the Polytopes regularly Derived from the Regular Polytopes," Section I: The Simplex, *Verhandelingen* of Amsterdam, Vol. XI (1911), No. 3.

Sections II, III, IV (Measure Polytope, Cross Polytope, Half-measure Polytope) will be published this year. The last section (on Extra Polytopes) will appear later on.

either *prisms* or *prismotopes** or both, whilst there is no simplex net admitting either a prism or a prismotope. Now the four-dimensional angles of prism and prismotope are known; these angles will be tabulated — with indication of the source from which they are taken — at the place where we want them, *i. e.*, in § 4.

Before entering on our task, we have still to remark that for our aim it is not sufficient to know the different kinds of constituents of any net, but that we must know also how many constituents of each kind concur in any vertex of the net. This *multiplicity* of each constituent, which is independent of the choice of the vertex, has been determined in another paper. †

§ 2. *The Simplex Group.*

The simplex group contains 15 polytopes, the simplex itself included; they are represented in the following table by their expansion symbol, the coordinates of their vertices and their characteristic numbers of vertices, edges, faces, limiting polyhedra. Instead of C_5 = five-cell, we use henceforth the more appropriate symbol $S(5)$ = simplex with five vertices.

$S(5)$	(1 0 0 0 0)	(5, 10, 10, 5)	
e_1 "	(2 1 0 0 0)	(20, 40, 30, 10)	
e_2 "	(2 1 1 0 0)	(30, 90, 80, 20)	
e_3 "	*(2 1 1 1 0)	(20, 60, 70, 30)	
$e_1 e_2$ "	(3 2 1 0 0)	(60, 120, 80, 20)	
$e_1 e_3$ "	(3 2 1 1 0)	(60, 150, 120, 30)	
$e_2 e_3$ "	(3 2 2 1 0)	(60, 150, 120, 30)	$= - e_1 e_3 S(5)$
$e_1 e_2 e_3$ "	*(4 3 2 1 0)	(120, 240, 150, 30)	
$c e_1$ "	(1 1 0 0 0)	(10, 30, 30, 10)	
$c e_2$ "	(1 1 1 0 0)	(10, 30, 30, 10)	$= - c e_1 S(5)$
$c e_3$ "	(1 1 1 1 0)	(5, 10, 10, 5)	$= - S(5)$
$c e_1 e_2$ "	*(2 2 1 0 0)	(30, 60, 40, 10)	
$c e_1 e_3$ "	(2 2 1 1 0)	(30, 90, 80, 20)	$= - e_2 S(5)$
$c e_2 e_3$ "	(2 2 2 1 0)	(20, 40, 30, 10)	$= - e_1 S(5)$
$c e_1 e_2 e_3$ "	(3 3 2 1 0)	(60, 120, 80, 20)	$= - e_1 e_2 S(5)$

Here (3 2 1 0 0) means that we get the coordinates of the 60 vertices of the polytope $e_1 e_2 S(5)$ with respect to a regular five-cell by assigning to x_1, x_2, x_3, x_4, x_5 the numerical values 3, 2, 1, 0, 0 taken successively in all the possible

* See the definition of the prismotope $(m; n)$, § 4, and compare, for prismotopes in general: "The Characteristic Numbers of the Prismotope," *Proceedings* of Amsterdam, Vol. XIV (1911), p. 424.

† "On Reciprocal Nets," *Nieuw Archief voor Wiskunde*, Vol. X (1913), p. 273.

orders of succession, whilst the remark $e_2 e_3 S(5) = -e_1 e_3 S(5)$ indicates that $e_1 e_3 S(5)$ and $e_2 e_3 S(5)$ are congruent but of opposite orientation. Finally, the asterisk before the coordinate symbol of a polytope expresses the fact that the latter is central symmetric.

It follows from this table that we have to determine nine four-dimensional angles. But in the first paper on four-dimensional angles* we found for the four-dimensional angle A of $S(5)$ itself the value $-\frac{16}{5} + 4\varepsilon$, where $\varepsilon = 75° \, 31' \, 21''$, *i. e.*, $\cos\varepsilon = \frac{1}{4}$, is the angle between the spaces of any two limiting tetrahedra of the simplex. So the number of unknown angles is reduced to *eight*. On the other hand, there are only *seven* simplex nets; so, even if the relations deduced from these nets are mutually independent — which supposition will prove to be too optimistic — we want one relation more. We start by indicating a way leading to such a relation.

By means of a regular truncation of $S(5)$ at the vertices so as to take away a third part of each edge on either side, we get the polytope $e_1 S(5)$. So we get the solid angle round an edge of the original $S(5)$ by addition of the four-dimensional angles of the $e_1 S(5)$ and that of the smaller $S(5)$ taken away by truncation, *i. e.*, by the generally known rule of the spherical excess† we find $A_1 + A = 2(3\varepsilon - 2)$, if A_1 stands for the four-dimensional angle of $e_1 S(5)$. So we find $A_1 = -\frac{4}{5} + 2\varepsilon$.

The seven simplex nets are characterized in the following table by the constituents concurring in any vertex:

I	$10\,S(5)$,	$20\,c\,e_1 S(5)$	
II	$2\,S(5)$,	$8\,e_1 S(5)$,	$6\,c\,e_1 e_2 S(5)$
III	$2\,c\,e_1 S(5)$,	$6\,e_2 S(5)$,	$2\,e_3 S(5)$
IV	$2\,e_1 S(5)$,	$6\,e_1 e_2 S(5)$,	$e_3 S(5)$
V	$2\,e_2 S(5)$,	$3\,e_1 e_3 S(5)$,	$2\,c\,e_1 e_2 S(5)$
VI	$2\,e_1 e_2 S(5)$,	$2\,e_1 e_2 e_3 S(5)$,	$2\,e_1 e_3 S(5)$
VII	$5\,e_1 e_2 e_3 S(5)$		

So, if the four-dimensional angles of $e_i e_k \ldots S(5)$ and $c\,e_i e_k \ldots S(5)$ are indicated by $A_{ik\ldots}$ and $A'_{ik\ldots}$ respectively, we find the equations

$$10A + 20A'_1 = 2A + 8A_1 + 6A'_{12} = 2A'_1 + 6A_2 + 2A_3 = 2A_1 + 6A_{12} + A_3$$
$$= 2A_2 + 3A_{13} + 2A'_{12} = 2A_{12} + 2A_{123} + 2A_{13} = 5A_{123} = 16.$$

* Henceforth this study will be quoted as "F. P."

† This angle $3\varepsilon - 2$ occurs already in the small table of "F. P." in the row of C_8; it is the angle $\beta = 46° \, 34' \, 3''$.

These relations allow us to complete the two results already obtained by

$$A_{123} = \frac{16}{5}, \quad A'_1 = A'_2 = \frac{12}{5} - 2\,\varepsilon, \quad A'_{12} = \frac{24}{5} - 4\,\varepsilon,$$

but the four equations

$$\left.\begin{array}{rl} \frac{24}{5} - 4\,\varepsilon + 6\,A_2 \; + 2\,A_3 & = 16, \\ -\frac{8}{5} + 4\,\varepsilon + 6\,A_{12} \; + \; A_3 & = 16, \\ \frac{24}{5} - 4\,\varepsilon + 2\,A_2 \; + 4\,A_{13} & = 16, \\ \frac{32}{5} \qquad\quad + 2\,A_{12} + 2\,A_{13} & = 16, \end{array}\right\} \qquad (1)$$

with the remaining four unknown angles A_2, A_3, A_{12}, A_{13}, prove to admit one degree of dependence. So we still want another relation from a new source.

§ 3. *The Simplex Group Continued. A New Equation.*

We try to determine one of the four unknown angles figuring in the equations (1) by itself, and fix our choice on A_{12}, as this angle is formed by four edges, whilst five edges concur in the vertex of A_{13} and even six in the vertices of A_2 and A_3.

If the point $3, 2, 1, 0, 0$ is chosen as vertex O of angle A_{12}, the four adjacent points are

$$P = 2, 3, 1, 0, 0\,, \quad Q = 3, 1, 2, 0, 0\,, \quad R_1 = 3, 2, 0, 1, 0\,, \quad R_2 = 3, 2, 0, 0, 1\,.$$

So we have to determine the six plane angles POQ, POR_i, QOR_i, R_1OR_2 by means of the mutual distances of the points O, P, Q, R_i. We find

$$OP = OQ = OR_i = R_1R_2 = \sqrt{2}, \quad PQ = QR_i = \sqrt{6}, \quad PR_i = 2\,;$$

and therefore

$$POQ = QOR_i = 120^\circ, \quad POR_i = 90^\circ, \quad R_1OR_2 = 60^\circ.$$

We have represented the angle $O\,(PQR_1R_2) = x$ on the spherical space (Fig. 1) with O as center, and introduce in this diagram three more points:

In the first place, the point Q' diametrically opposite to Q, giving rise to the formation of the new angle $O\,(PQ'R_1R_2) = y$, forming with $O\,(PQR_1R_2)$ the solid angle on the edge QQ' of the tangents in Q' to the arcs $Q'P$, $Q'R_1$, $Q'R_2$.

In the second place, the point P' diametrically opposite to P, and in the third place the point Q'' bisecting the arc $Q'P'$, giving rise to the formation of two new angles, *i. e.*, $O\,(Q'Q''R_1R_2) = A$, as all the six plane angles are 60°, and

$O(Q'' P' R_1 R_2) = y$. So $2y + A$ represents* the solid angle on the edge PP' of the tangents in P to the arcs PQ', PR_1, PR_2.

So we find

$$x + y = 2\,\text{sph. exc.}\, Q'\,(P R_1 R_2), \quad 2y + A = 2\,\text{sph. exc.}\, P\,(Q' R_1 R_2),$$

and therefore

$$x = \frac{1}{2} A + 2\,\text{sph. exc.}\, Q'\,(P R_1 R_2) - \text{sph. exc.}\, P\,(Q' R_1 R_2). \qquad (2)$$

Calculation of sph. exc. $Q'\,(P R_1 R_2)$. The corresponding spherical triangle admits as *sides* one angle of an equilateral triangle with sides of $60°$, and two angles equal to the vertex angle of an isosceles triangle with sides $60°, 60°, 90°$; so we get for these sides once $\cos^{-1}\frac{1}{3}$ and twice $\cos^{-1} - \frac{1}{3}$, *i.e.*, δ, $2-\delta$, $2-\delta$, where $\delta = 70°\,31'\,44''$ is the angle between any two faces of a tetrahedron.

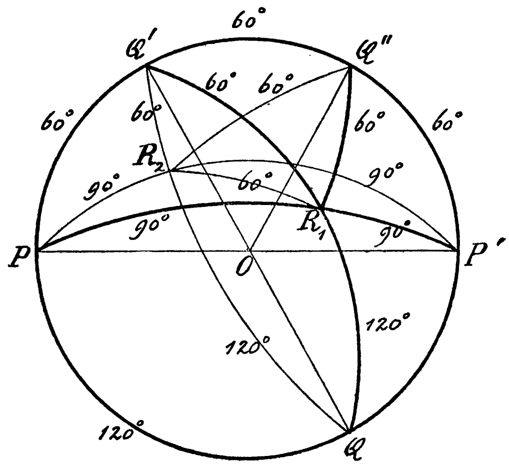

FIG. 1.

From the sides δ, $2-\delta$, $2-\delta$ we have to deduce the *angles*. Here we find once $\cos^{-1}\frac{1}{4}$ and twice $\cos^{-1} - \frac{1}{4}$, *i. e.*, ε, $2-\varepsilon$, $2-\varepsilon$, where $\varepsilon = 75°\,31'\,21''$ is the angle introduced in § 2. So the spherical excess of $Q'\,(P R_1 R_2)$ is $2-\varepsilon$.

Calculation of sph. exc. $P\,(Q' R_1 R_2)$. For the *sides* we find one angle of $60°$ (from triangle $P R_1 R_2$, where $P R_1$ and $P R_2$ are right) and two angles equal to the base angle of the isosceles triangle with sides $60°, 60°, 90°$; so the sides are once $60°$ and twice $\cos^{-1}\frac{1}{3}\sqrt{3}$.

* We owe the ingenious determination of the angle y to Dr. W. A. WYTHOFF, of Amsterdam, whose dissertation "De Biquaternion als Bewerking in de Ruimte van vier Afmetingen" (The Biquaternion as Operation in Four-dimensional Space), Amsterdam, 1898, forms an excellent introduction to the geometry of S_4.

Passing from the sides to the *angles*, we get once the angle ε and twice $\cos^{-1}\frac{1}{6}\sqrt{6} = \frac{1}{2}\cos^{-1} - \frac{2}{3}$. So, if we represent $\cos^{-1} - \frac{2}{3}$ by ϕ, the spherical excess of $P\,(Q'R_1R_2)$ is $\varepsilon + \phi - 2$. As $\cos\,(\varepsilon + \phi - 180°) = \frac{1}{12}(2 + 5\sqrt{3})$, we use the result $\cos^{-1}\frac{1}{12}(2+5\sqrt{3})$.

Calculation of x. By introducing the value $-\frac{8}{5} + 2\,\varepsilon$ of $\frac{1}{2}A$ and those of the spherical excesses, (2) passes into

$$x = \frac{12}{5} - \cos^{-1}\frac{1}{12}(2+5\sqrt{3}),$$

which leads to the following evaluation:

$$2+5\sqrt{3} = 10{,}6602540, \qquad \log\ \frac{1}{12}(2+5\sqrt{3}) = 9{,}9485863,$$

$$\frac{1}{12}(2+5\sqrt{3}) = 0{,}8883545, \qquad \cos^{-1}\frac{1}{12}(2+5\sqrt{3}) = 27°\,19'\,58''.$$

We denote the latter angle by ε'. Then $x = \frac{12}{5} - \varepsilon'$.

So we find by means of the equations (1)

$$A_2 = \frac{52}{15} + 2\,\varepsilon - 2\,\varepsilon', \quad A_3 = \frac{16}{5} - 4\,\varepsilon + 6\,\varepsilon', \quad A_{13} = \frac{12}{5} + \varepsilon'$$

by introducing $A_2 = \frac{12}{5} - \varepsilon'$.

These results are tabulated at the end of this paper.

§ 4. *Angles of Four-dimensional Prisms and Prismotopes.*

We find in the measure-polytope nets, prisms on C, O, CO, RCO, tC, tO, tCO and prismotopes $(4;4)$, $(4;8)$, $(8;8)$; in the nets derived from the cell C_{24}, prisms on T, C, O, CO, RCO, tT, tC, tO, tCO and prismotopes $(3;3)$, $(3;6)$, $(6;6)$. As the four-dimensional angle of any of these prisms expressed in right angles is equal to the solid angle of the base of the prism expressed in right angles, these angles can be taken from A. ANDREINI's tables.* As the prismotope $(m;n)$ is generated by placing two regular polygons p_m and p_n in such a way in two planes perfectly normal to each other that the point common to these planes is a common vertex of the polygons, and moving

* "Sulle reti di poliedri regolari e semiregolari e sulle correspondenti reti correlative" (On the Nets of Regular and Semiregular Polyhedra and on the Corresponding Reciprocal Nets), *Memorie* della Società italiana delle scienze (detta dei XL), 3d series, Vol. XIV (1905), p. 75.

one of them so as to bring that vertex of it successively into coincidence with any point in the interior of the other, the four-dimensional angle of that prismotope expressed in right angles is the product of the plane angles of p_m and p_n expressed in right angles. So we find, if P_T stands for "prism on T" and AP_T and $A\ (m;\ n)$ for the four-dimensional angles of P_T and $(m;\ n)$, whilst $\delta = 70^\circ\ 31'\ 44''$ is the angle introduced in § 3:

$AP_T = -2 + 3\delta$	$AP_{CO} = 2\delta$	$AP_{tC} = 1 + \delta$	$A\ (4;4) = 1$	$A\ (3;3) = \frac{4}{9}$
$AP_C = 1$	$AP_{RCO} = 3 - \delta$	$AP_{tO} = 2$	$A\ (4;8) = \frac{3}{2}$	$A\ (3;6) = \frac{8}{9}$
$AP_O = 4 - 4\delta$	$AP_{tT} = 2 - \delta$	$AP_{tCO} = \frac{5}{2}$	$A\ (8;8) = \frac{9}{4}$	$A\ (6;6) = \frac{16}{9}$

§ 5. *The Measure-Polytope Group.*

The 15 polytopes of this group, the measure polytope C_8 itself included, are represented in the following table by their expansion symbol, the coordinates of their vertices and their characteristic numbers;* moreover, a fourth column gives the interpretation of these polytopes as derived from the cross polytope C_{16}.

C_8	[1 1 1 1]	(16, 32, 24, 8)	$c\,e_3\,C_{16}$
e_1 "	[1′ 1′ 1′ 1]	(64, 128, 88, 24)	$c\,e_2\,e_3$ "
e_2 "	[1′ 1′ 1 1]	(96, 288, 248, 56)	$c\,e_1\,e_3$ "
e_3 "	[1′ 1 1 1]	(64, 192, 208, 80)	e_3 "
$e_1\,e_2$ "	[2′ 2′ 1′ 1]	(192, 384, 248, 56)	$c\,e_1\,e_2\,e_3$ "
$e_1\,e_3$ "	[2′ 1′ 1′ 1]	(192, 480, 368, 80)	$e_2\,e_3$ "
$e_2\,e_3$ "	[2′ 1′ 1 1]	(192, 480, 368, 80)	$e_1\,e_3$ "
$e_1\,e_2\,e_3$ "	[3′ 2′ 1′ 1]	(384, 768, 464, 80)	$e_1\,e_2\,e_3$ "
$c\,e_1$ "	[1 1 1 0] $\sqrt{2}$	(32, 96, 88, 24)	$c\,e_2$ "
$c\,e_2$ "	[1 1 0 0] $\sqrt{2}$	(24, 96, 96, 24)	$c\,e_1$ " $= C_{24}$
$c\,e_3$ "	[1 0 0 0] $\sqrt{2}$	(8, 24, 32, 16)	C_{16}
$c\,e_1\,e_2$ "	[2 2 1 0] $\sqrt{2}$	(96, 192, 120, 24)	$c\,e_1\,e_2$ "
$c\,e_1\,e_3$ "	[2 1 1 0] $\sqrt{2}$	(96, 288, 240, 48)	e_2 "
$c\,e_2\,e_3$ "	[2 1 0 0] $\sqrt{2}$	(48, 120, 96, 24)	e_1 "
$c\,e_1\,e_2\,e_3$ "	[3 2 1 0] $\sqrt{2}$	(192, 384, 240, 48)	$e_1\,e_2$ "

Here [2 2 1 0] $\sqrt{2}$ means that we obtain the coordinates of the 96 vertices of $c\,e_1\,e_2\,C_8$ with respect to a system of rectangular axes by assigning to $x_1,\ x_2,\ x_3,\ x_4$ the numerical values $2\sqrt{2},\ 2\sqrt{2},\ \sqrt{2},\ 0$ taken successively in

* Compare for these data and for the measure-polytope nets: "Analytical Treatment," etc., Section II.

all the possible orders of succession with all the possible combinations of the positive and negative sign; moreover, $1'$, $2'$, $3'$ stand for short for $1+\sqrt{2}$, $1+2\sqrt{2}$, $1+3\sqrt{2}$.

As the four-dimensional angles of C_8, C_{16}, C_{24} are known (see the table in "F. P."), we have still to determine 12 angles. To that end we have at our disposal the following sixteen nets, characterized by the constituents concurring in any vertex:

I	$8\,e_1 C_8$	....	....	....	$c\,e_3 C_8 = C_{16}$
II	$e_1 C_8$	$3\,P_{tC}$	$3\,(4;8)$	P_C	$e_3 C_8$
III	$4\,e_2 C_8$	....	....	$2\,P_O$	$c\,e_2 C_8 = C_{24}$
*IV	$e_2 C_8$	$2\,P_{RCO}$	$(4;4)$	$2\,P_{RCO}$	$e_2 C_8$
V	$2\,e_3 C_8$	....	$3\,(4;4)$	$3\,P_{CO}$	$c\,e_1 C_8$
VI	$4\,e_1 e_2 C_8$	....	....	P_O	$c\,e_2 e_3 C_8$
VII	$e_1 e_2 C_8$	$2\,P_{tCO}$	$(4;8)$	P_{RCO}	$e_2 e_3 C_8$
VIII	$2\,e_1 e_3 C_8$	....	$2\,(4;8)$	P_{CO}	$c\,e_1 e_2 C_8$
*IX	$e_1 e_3 C_8$	P_{tC}	$2\,(8;8)$	P_{tC}	$e_1 e_3 C_8$
X	$2\,e_2 e_3 C_8$	....	$(4;4)$	$2\,P_{tO}$	$c\,e_1 e_2 C_8$
XI	$2\,e_1 e_2 e_3 C_8$	....	$(4;8)$	P_{tO}	$c\,e_1 e_2 e_3 C_8$
*XII	$e_1 e_2 e_3 C_8$	P_{tCO}	$(8;8)$	P_{tCO}	$e_1 e_2 e_3 C_8$
XIII	$8\,c\,e_1 C_8$	....	....	....	$2\,c\,e_3 C_8 = 2\,C_{16}$
XIV	$4\,c\,e_1 e_2 C_8$	....	....	....	$2\,c\,e_2 e_3 C_8$
*XV	$2\,c\,e_1 e_3 C_8$	....	$2\,(4;4)$	....	$2\,c\,e_1 e_3 C_8$
*XVI	$2\,c\,e_1 e_2 e_3 C_8$	....	$(4;4)$	....	$2\,c\,e_1 e_2 e_3 C_8$

Here the asterisk before the rank number indicates that the net is symmetric.

Now, if we represent the angles of $e_i e_k \ldots C_8$ and $c\,e_i e_k \ldots C_8$ by $B_{ik\ldots}$ and $B'_{ik\ldots}$ respectively, and indicate the sum of the angles of the constituents concurring in a vertex by an S with the rank of the net as subscript, we get

$$
\begin{array}{ll|ll}
S_1 = 8\,B_1 + B'_3 & & S_9 = 2\,B_{13} + \frac{13}{2} + 2\,\delta & \\
S_2 = B_1 + B_3 + \frac{17}{2} + 3\,\delta & & S_{10} = 2\,B_{23} + B'_{12} + 5 & \\
S_3 = 4\,B_2 + B'_2 + 8 - 8\,\delta & & S_{11} = 2\,B_{123} + B'_{123} + \frac{7}{2} & \\
S_4 = 2\,B_2 + 13 - 4\,\delta & & S_{12} = 2\,B_{123} + \frac{29}{4} & \\
S_5 = 2\,B_3 + B'_1 + 3 + 6\,\delta & & S_{13} = 8\,B'_1 + 2\,B'_3 & \\
S_6 = 4\,B_{12} + B'_{23} + 4 - 4\,\delta & & S_{14} = 4\,B'_{12} + 2\,B'_{23} & \\
S_7 = B_{12} + B_{23} + \frac{19}{2} - \delta & & S_{15} = 4\,B_{13} + 2 & \\
S_8 - 2\,B_{13} + B'_{13} + 3 + 2\,\delta & & S_{16} = 4\,B'_{123} + 1 &
\end{array}
$$

So we find, as we have already $B'_2 = 2$, $B'_3 = \frac{2}{3}$, from $S_i = 16$:

$(S_1)\ldots B_1 = \frac{23}{12}$	$(S_9)\ldots B_{13} = \frac{19}{2} - \delta$
$(S_2)\ldots B_3 = \frac{67}{12} - 3\,\delta$	(either S_8 or S_{15}) $\ldots B'_{13} = \frac{7}{2}$
(either S_3 or S_4) $\ldots B_2 = \frac{3}{2} + 2\,\delta$	$(S_{12})\ldots B_{123} = \frac{35}{6}$
(" S_5 " S_{13}) $\ldots B'_1 = \frac{11}{6}$	(" S_{11} " S_{16}) $\ldots B'_{123} = \frac{15}{4}$

and the now remaining four equations

$$\left.\begin{aligned} 4\,B_{12} + B'_{23} + 4 - 4\,\delta &= 16, \\ B_{12} + B_{23} + \frac{19}{2} - \delta &= 16, \\ 2\,B_{23} + B'_{12} + 5 &= 16, \\ 4\,B'_{12} + 2\,B'_{23} &= 16, \end{aligned}\right\} \tag{3}$$

with the four unknown angles B_{12}, B'_{12}, B_{23}, B'_{23}, prove to admit one degree of dependence. So here also we want another relation from a new source.

By means of a regular truncation of C_{16} at the vertices, so as to take away a third part of each edge on either side, we get the polytope $e_1 C_{16}$. As the four-dimensional angle taken away by truncation at any vertex of this $e_1 C_{16}$ is evidently half that angle of C_{16}, we get the solid angle round an edge of C_{16} by addition of the four-dimensional angle of $e_1 C_{16}$ and half that angle of C_{16}. Now the latter angle (see "F. P." in the table, the angle β of C_{16}) is $\frac{4}{3}$; as $e_1 C_{16} = c\,e_2\,e_3\,C_8$, we find therefore $B'_{23} + \frac{1}{2} B'_3 = \frac{8}{3}$, *i. e.*, $B'_{23} = \frac{7}{3}$. Then the equations (3) give $B_{12} = \frac{29}{12} + \delta$, $B_{23} = \frac{49}{12}$, $B'_{12} = \frac{17}{6}$.

These results are tabulated at the end of this paper, both ways (for C_8 and C_{16}).

§ 6. *The Group of the Cell C_{24}.*

The 15 polytopes of this group, the cell C_{24} itself included, are represented in the following table by their expansion symbol and their characteristic number; here we omit the coordinates of their vertices, as some of these polytopes require two or even three coordinate symbols,* but a third column gives their interpretation as derived either from C_8 or from C_{16} or in a different way from C_{24}.

C_{24}	(24, 96, 96, 24)	
e_1 "	(192, 384, 240, 48)	$= c\,e_1\,e_2\,e_3\,C_8 = e_1\,e_2\,C_{16}$
e_2 "	(288, 864, 720, 144)	
e_3 "	(144, 576, 672, 240)	
$e_1\,e_2$ "	(576, 1152, 720, 144)	
$e_1\,e_3$ "	(576, 1440, 1104, 240)	
$e_2\,e_3$ "	(576, 1440, 1104, 240)	$= e_1\,e_3\,C_{24}$
$e_1\,e_2\,e_3$ "	(1152, 2304, 1392, 240)	
$c\,e_1$ "	(96, 288, 240, 48)	$= c\,e_1\,e_3\,C_8 = e_2\,C_{16}$
$c\,e_2$ "	(96, 288, 240, 48)	$= c\,e_1\,e_3\,C_8 = e_2\,C_{16} = c\,e_1\,C_{24}$
$c\,e_3$ "	(24, 96, 96, 24)	$= C_{4}$
$c\,e_1\,e_2$ "	(288, 576, 336, 48)	
$c\,e_1\,e_3$ "	(288, 864, 720, 144)	$= e_2\,C_{24}$
$c\,e_2\,e_3$ "	(192, 384, 240, 48)	$= e_1\,C_{24}$
$c\,e_1\,e_2\,e_3$ "	(576, 1152, 720, 144)	$= e_1\,e_2\,C_{24}$

It follows from this table that we have still to determine six angles only, viz., those of $e_2\,C_{24}$, $e_3\,C_{24}$, $e_1\,e_2\,C_{24}$, $e_1\,e_3\,C_{24}$, $e_1\,e_2\,e_3\,C_{24}$, $c\,e_1\,e_2\,C_{24}$, which angles will be denoted by C_2, C_3, C_{12}, C_{13}, C_{123}, C'_{12}. Now we have at our disposal 29 semiregular nets each of which has for constituents, besides prism and prismotope, one measure-polytope form and one form derived from cell C_{24}, *i. e.*, never more than *one* unknown angle. So we can abridge the work by mentioning only those six nets which contain the angles enumerated above. They are, once more with the constituents concurring in any vertex:

I	$c\,e_1\,C_8$	$2\,P_T$	....	$3\,e_2\,C_{24}$
II	$c\,e_2\,C_8$	$4\,P_O$	4 (3 ; 3)	$2\,e_3\,C_{24}$
III	$e_1\,C_8$	P_T	....	$3\,e_1\,e_2\,C_{24}$
IV	$e_2\,C_8$	P_O	2 (3 ; 6)	$2\,e_1\,e_3\,C_{24}$
V	$e_1\,e_2\,C_8$	P_{tT}	(6 ; 3)	$2\,e_1\,e_2\,e_3\,C_{24}$
VI	$2\,e_1\,C_8$	....	....	$3\,c\,e_1\,e_2\,C_{24}$

So we find the six equations

* These symbols will be discussed in "Analytical Treatment, etc.," Section V. However, they figure already in Table VII, belonging to Section III.

$$\left.\begin{aligned}
3C_2 &= 16 - \frac{11}{6} - 2(-2+3\delta),\\
2C_3 &= 16 - 2 - 16(1-\delta) - \frac{16}{9},\\
3C_{12} &= 16 - \frac{23}{12} - (-2+3\delta),\\
2C_{13} &= 16 - \left(\frac{3}{2}+2\delta\right) - 4(1-\delta) - \frac{16}{9},\\
2C_{123} &= 16 - \left(\frac{29}{12}+\delta\right) - (2-\delta) - \frac{8}{9},\\
3C'_{12} &= 16 - \frac{23}{6},
\end{aligned}\right\}$$

giving

$$C_2 = \frac{109}{18} - 2\delta, \quad C_{12} = \frac{193}{36} - \delta, \quad C_{123} = \frac{385}{72},$$
$$C_3 = -\frac{17}{9} + 8\delta, \quad C_{13} = \frac{157}{36} + \delta, \quad C'_{12} = \frac{73}{18}$$

§ 7. *Results.*

The results are put on record in the following table:

	C_5		C_8		C_{16}		C_{24}	
1	$-\frac{16}{5}+4\varepsilon$	14° 5′ 4″	1	90°	$\frac{2}{3}$	120°	2	180°
e_1	$-\frac{4}{5}+2\varepsilon$	79 2 42	$\frac{23}{12}$	172 30′	$\frac{7}{3}$	210	$\frac{15}{4}$	337 30′
e_2	$\frac{4}{5}+2\varepsilon-2\varepsilon'$	168 22 46	$\frac{3}{2}+2\delta$	141 3 27″	$\frac{7}{2}$	315	$\frac{109}{18}-2\delta$	403 56 33″
e_3	$\frac{16}{5}-4\varepsilon+6\varepsilon'$	149 54 24	$\frac{67}{12}-3\delta$	290 54 49	$\frac{67}{12}-3\delta$	290 54′ 49″	$-\frac{17}{9}+8\delta$	94 13 49
$e_1 e_2$	$\frac{12}{5}-\varepsilon'$	188 40 2	$\frac{29}{12}+\delta$	287 1 44	$\frac{15}{4}$	337 30	$\frac{193}{36}-\delta$	411 58 16
$e_1 e_3$	$\frac{12}{5}+\varepsilon'$	243 19 58	$\frac{19}{4}-\delta$	351 58 16	$\frac{49}{12}$	367 30	$\frac{157}{36}+\delta$	463 1 44
$e_2 e_3$	$\frac{12}{5}+\varepsilon'$	243 19 58	$\frac{49}{12}$	367 30	$\frac{19}{4}-\delta$	351 58 16	$\frac{157}{36}+\delta$	463 1 44
$e_1 e_2 e_3$	$\frac{16}{5}$	288	$\frac{35}{8}$	393 45	$\frac{35}{8}$	393 45	$\frac{385}{72}$	481 15
$c\,e_1$	$\frac{12}{5}-2\varepsilon$	64 57 18	$\frac{11}{6}$	165	2	180	$\frac{7}{2}$	315
$c\,e_2$	$\frac{12}{5}-2\varepsilon$	64 57 18	2	180	$\frac{11}{6}$	165	$\frac{7}{2}$	315
$c\,e_3$	$-\frac{16}{5}+4\varepsilon$	14 5 24	$\frac{2}{3}$	120	1	90	2	180
$c\,e_1 e_2$	$\frac{24}{5}-4\varepsilon$	129 54 36	$\frac{17}{6}$	255	$\frac{17}{6}$	255	$\frac{73}{18}$	365
$c\,e_1 e_3$	$\frac{4}{5}+2\varepsilon-2\varepsilon'$	168 22 46	$\frac{7}{2}$	315	$\frac{3}{2}+2\delta$	141 3 27	$\frac{109}{18}-2\delta$	403 56 33
$c\,e_2 e_3$	$-\frac{4}{5}+2\varepsilon$	79 2 42	$\frac{7}{3}$	210	$\frac{23}{12}$	172 30	$\frac{15}{4}$	337 30
$c\,e_1 e_2 e_3$	$\frac{12}{5}-\varepsilon'$	188 40 2	$\frac{15}{4}$	337 30	$\frac{29}{12}+\delta$	287 1 44	$\frac{193}{36}-\delta$	411 58 16

$\delta = 70^\circ\,31'\,43'', 624; \quad \varepsilon = 75^\circ\,31'\,20'', 951; \quad \varepsilon' = 27^\circ\,19'\,58'', 037.$

These results are entirely in accordance with the laws of reciprocity,* of C_5 with itself, of C_8 and C_{16} with each other, of C_{24} with itself.

GRONINGEN, *January* 12, 1913.

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* Compare "Reciprocity in Connection with Semiregular Polytopes and Nets," *Proceedings* of Amsterdam, Vol. XIII (1910), p. 384.

As C_{120} and C_{600} are polarly related, the offspring of the one is equal to that of the other; so the law of reciprocity will require the equality of the angles of $e_1 C_{120}$ and $c\, e_1\, e_3\, C_{600}$, etc.

It may be useful to remember that the Mathematical Society of Amsterdam had proposed the following prize question:

"To determine the four-dimensional angles of the semiregular polytopes of space S_4 by a direct method, and to deduce from it all the possible space fillings by polytopes with one kind of vertex and one length of edge."

This question has been withdrawn, as all the nets are found. Nevertheless it still remains a desideratum: to determine directly the angles of the polytopes belonging to C_{120} and C_{600}.