Math. Z. 200, 3–45 (1988)

Mathematische Zeitschrift

Regular and Semi-Regular Polytopes. III

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Contents

3.1. Introduction

As was promised in the preface to Part I in this *Zeitschrift* **46**, 380–407 (1940), this third and final 'Part' deals mainly with uniform polytopes in six, seven and eight dimensions. But let us begin with a brief recapitulation of Parts I and II and a summary of the present Part III.

A polygon is said to be *uniform* if it is regular. A convex polytope is said to be uniform if its facets (or cells) are uniform and its symmetry group is transitive on the vertices. The uniform polyhedra are the Platonic and Archimedean solids along with the regular prisms and antiprisms. A uniform polytope in 4 dimensions is determined by its *vertex figure*: the section by a hyperplane through points at unit distance from one vertex A along all the edges issuing from A. This is a polyhedron Π having, for each p-gon at A, an edge of length $2\cos\pi/p$. The faces of Π are, of course, vertex figures of uniform polyhedra [25, p. 434]. By examining all such polyhedra Π inscribed in spheres with radius less than 1, J.H. Conway made a complete list of convex uniform polytopes in 4 dimensions, with the conclusion that all but one of them can be obtained by means of a construction first used by W.A. Wythoff. The set of vertices is the orbit of a suitable point for one of the finite reflection groups ('Weyl groups') or for the rotatory subgroup of such a group. This construction suggests a systematic notation for the polytopes, which was developed in Parts I and II.

The idea of representing a reflection group by a graph of 'dots and links' occurred to me in 1931 [12, p. 133; 14, p. 619]. Ten years later, Witt [57, p. 301] proposed a slightly modified version of the graph, replacing a link marked q by an unmarked $(q-2)$-fold link (which agrees neatly with the absence of a link when $q=2$). This 'Dynkin diagram' has been found useful for many purposes [26, pp. 519–547; 54]. It was quite independently rediscovered by Dynkin [29; 30], suggesting the possibility that the best ideas (such as non-Euclidean geometry) are not merely invented but somehow lying in wait to be revealed.

The existence of Conway's 'grand antiprism' (§ 2.8) indicates that spaces of more than four dimensions may be expected to admit many uniform polytopes whose symmetry groups are not generated by reflections. Their classification will provide an interesting task for future geometers. Part III is restricted to polytopes derived from reflection groups by Wythoff's construction, and especially to the polytopes p_{qr} $(=p_{rq})$ which come from the 'sporadic' Weyl groups E_n $(n=p+q+r+1)$.

As we saw in § 2.2 (on page 563 of Part II), the fundamental region for the symmetry group of a *regular* polytope is a spherical *orthoscheme*: a simplex $ABCD\ldots$ whose 'successive' edges AB, BC, $CD, \ldots$ are mutually perpendicular. In § 3.5 we shall see how the fundamental region for each E_n $(n\leqq 9)$ can be dissected into three orthoschemes, two of which are oppositely congruent. Their contents are expressed in terms of Schläfli functions, providing an interesting formula for the order of E_n as a function of n.

We shall see in § 3.6 that the three polytopes p_{qr}, q_{rp}, r_{pq} exhibit a relation of *triality*, analogous to the *duality* that relates two reciprocal regular polytopes

$$\{p, q, \ldots, w\} \quad \text{and} \quad \{w, \ldots, q, p\}.$$

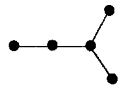
§ 3.7 and § 3.8 include descriptions of the correspondence between the polytopes 2_{31}, 1_{42} and certain configurations discovered by Hirschfeld and Longuet-Higgins, respectively. Finally, in § 3.9 we will discuss some properties of the so-called 'Coxeter transformation' $\mathrm{R}_1 \mathrm{R}_2 \ldots \mathrm{R}_n$.

3.2. The 'Classical' Reflection Groups

In Chapter XI of *Regular Polytopes* [21, pp. 188–196], reducible reflection groups were seen to be direct products of irreducible groups, each generated by reflections in the facets of a *simplex*: spherical or Euclidean according as the group is finite or infinite. Such a simplicial kaleidoscope is represented by a graph of 'dots' and 'links', with a dot for each mirror (that is, for each facet of the simplex) and a link for each acute dihedral angle π/q, marked q whenever $q>3$.

The 4-dimensional kaleidoscopes (using 4 or 5 mirrors) were described in § 2.2 (on pp. 563–567). In 5 dimensions, the three finite groups ('Weyl groups') are

$[3^4]=A_5\cong\mathfrak{S}_6$

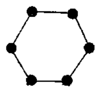


$[3^{2,1,1}]=D_5$

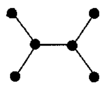
$[4, 3^3]=B_5\cong\mathfrak{C}_2\wr\mathfrak{S}_5$

with orders $6!$ $\quad 2^4 5!$ $\quad 2^5 5!$

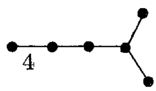
and the four infinite groups are



$[3^{[6]}]=\tilde{A}_5$



$[3^{1,1}, 3, 3^{1,1}]=\tilde{D}_5$



$[4, 3^2, 3^{1,1}]=\tilde{B}_5$

$[4, 3^3, 4]=\tilde{C}_5$

[6, p. 199; 34, pp. 57, 60]. Analogously when $n>5$, we have the finite groups

$$[3^{n-1}]=A_n\cong\mathfrak{S}_{n+1},\quad [3^{n-3,1,1}]=D_n,\quad [4, 3^{n-2}]=B_n\cong\mathfrak{C}_2\wr\mathfrak{S}_n$$

(see § 2.5, pp. 573–574), and the infinite groups

$$[3^{[n+1]}]=\tilde{A}_n,\quad [3^{1,1}, 3^{n-4}, 3^{1,1}]=\tilde{D}_n,\quad [4, 3^{n-3}, 3^{1,1}]=\tilde{B}_n,\quad [4, 3^{n-2}, 4]=\tilde{C}_n.$$

The treatment of $\tilde{A}_4$ in § 2.6 (page 578 of Part II) extends easily to $\tilde{A}_n$ [see also 26, pp. 151–153].

B_n is, of course, the complete symmetry group of the n-cube $\gamma_n=\{4, 3^{n-2}\}$ and of its reciprocal, the cross polytope $\{3^{n-2}, 4\}$ [21, p. 158]. For an n-cube of edge-length 2, the 2^n vertices have coordinates

$$(\pm 1, \pm 1, \ldots, \pm 1)$$

and the $2n$ bounding hyperplanes are $x_\nu=\pm 1$ ($\nu=1, 2, \ldots, n$). The sections by the sequence of hyperplanes

$$x_1+x_2+\ldots+x_n=\nu \quad \text{with} \quad \nu=n-2,\ n-3,\ n-4,\ n-5,\ n-6,\ldots$$

can be identified as truncations of the vertex figure $\alpha_{n-1}=\{3^{n-2}\}$ (see § 2.5 on pages 573–574), namely

$$\alpha_{n-1}=\mathrm{t}_0\,\alpha_{n-1},\quad \mathrm{t}_{0,1}\,\alpha_{n-1},\quad \mathrm{t}_1\,\alpha_{n-1},\quad \mathrm{t}_{1,2}\,\alpha_{n-1},\quad \mathrm{t}_2\,\alpha_{n-1},\ldots$$

[23, p. 127]. In particular, the section by $x_1+x_2+\ldots+x_n=n-2r$ is $\mathrm{t}_{r-1}\,\alpha_{n-1}=\left\{\begin{matrix}3^{r-1}\\3^{n-r-1}\end{matrix}\right\}$ [21, pp. 157, 239].

Extending the procedure of § 2.2 (pp. 565, 567), we see that D_n is a subgroup of index 2 in B_n, likewise $\tilde{D}_n$ in $\tilde{B}_n$, and $\tilde{B}_n$ in $\tilde{C}_n$. Since B_n has order $2^n n!$, D_n has order $2^{n-1}n!$. In fact, D_n is the symmetry group of the half-measure-polytope or 'hemi-cube'

$h\gamma_n$

whose vertices are *alternate* vertices of the n-cube γ_n. For instance,

$$h\gamma_2 = \alpha_1, \quad h\gamma_3 = \alpha_3, \quad h\gamma_4 = \beta_4$$

(see § 2.5 on page 574 and § 2.6 on page 581).

The n mirrors for D_n may conveniently be taken to have the equations

$$x_1 = x_2, \quad x_2 = x_3, \quad x_3 = x_4, \ldots, x_{n-1} = x_n,$$
$$x_1 + x_2 = 0,$$

so that a typical vertex of $h\gamma_n$ (given by $x_1 = x_2 = \ldots = x_n$ and $x_1 + x_2 \neq 0$) is (1, 1, ..., 1). The transpositions

$$(1\ 2), \quad (2\ 3), \quad (3\ 4), \quad \ldots, \quad (n-1\ n)$$

generate a symmetric subgroup $[3^{n-3,1,0}] = [3^{n-2}] = \mathfrak{S}_n$. The remaining generator transposes x_1 and x_2 while reversing the signs of *both*. Thus the 2^{n-1} vertices of $h\gamma_n$ (of edge $2\sqrt{2}$) are

$$(\pm 1, \pm 1, \ldots, \pm 1)$$

with an *even* number of minus signs. For the complementary $h\gamma_n$ with an *odd* number of minus signs, the typical vertex $(-1, 1, \ldots, 1)$ is given by $x_1 \neq x_2$ and $-x_1 = x_2 = x_3 = \ldots = x_n$.

By removing one or two dots from the ringed graph for $h\gamma_n$ we obtain symbols for its various faces and their groups of stability. Then (as in [21, p. 202] for 2_{21}) we can apply page 572 of Part II to find that $h\gamma_n$ has

$$2^{n-1} n!/n! = 2^{n-1} \text{ vertices}, \quad 2^{n-1} n!/2^2 (n-2)! = 2^{n-2} \binom{n}{2} \text{ edges},$$

$$2^{n-1} n!/(k+1)!\,(n-k-1)! = 2^{n-1} \binom{n}{k+1} \text{ simplexes } \alpha_k \quad (2 \leqq k \leqq n-1)$$

and

$$2^{n-1} n!/2^{k-1} k!\,(n-k)! = 2^{n-k} \binom{n}{k} \ h\gamma_k\text{'s} \quad (3 \leqq k \leqq n-1)$$

[11, p. 364]. (Since $\alpha_3 = h\gamma_3$, the 3-faces consist of

$$2^{n-1}\binom{n}{4} + 2^{n-3}\binom{n}{3} \text{ tetrahedra.)}$$

In particular, the facets consist of

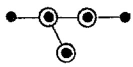
$$2^{n-1}\ \alpha_{n-1}\text{'s} \quad \text{and} \quad 2n\ h\gamma_{n-1}\text{'s}$$

corresponding to the 'remaining' vertices and the facets of γ_n.

Other uniform polytopes

$$\mathrm{h}_2\,\gamma_n,\quad \mathrm{h}_3\,\gamma_n,\quad \mathrm{h}_{2,3}\,\gamma_n,\ \ldots,\ \mathrm{h}_{2,3,\ldots,n-1}\,\gamma_n,$$

having the same symmetry group, are obtained by inserting extra rings. For instance, $\mathrm{h}_{2,3}\,\gamma_5$ is



Since the fundamental region satisfies the inequalities

$$x_1+x_2 \geqq 0, \quad x_1 \leqq x_2 \leqq x_3 \leqq x_4 \leqq x_5,$$

the $2^4\, 5!/2 \times 2 = 480$ vertices of this polytope are given by applying D_5 to the typical vertex

$$(1, 1, 3, 5, 5),$$

which comes from the equations

$$x_1 = x_2, \quad x_4 = x_5, \quad x_1+x_2 = x_3-x_2 = x_4-x_3.$$

We recall from § 2.5 (p. 575) that, when the dots representing $x_1 \pm x_2 = 0$ are *both* ringed, or *neither*, the polytope belongs to the family of β_n and γ_n.

It is interesting to observe that D_n is isomorphic to the group of automorphisms of certain configurations in Euclidean 3-space and the inversive plane [19, pp. 138–141; 20, pp. 258–262]. Let $\sigma_1, \sigma_2, \ldots$ be n planes through a point S, let $S_{\lambda\mu}$ be an arbitrary point on the line of intersection $\sigma_\lambda \cdot \sigma_\mu$, and let $\sigma_{\lambda\mu\nu}$ be the plane $S_{\lambda\mu} S_{\lambda\nu} S_{\mu\nu}$. Then the four planes $\sigma_{123}, \sigma_{124}, \sigma_{134}, \sigma_{234}$ all pass through one point, say S_{1234} (because $\sigma_1\,\sigma_2\,\sigma_3\,\sigma_{123}$ and $\sigma_{124}\,\sigma_{134}\,\sigma_{234}\,\sigma_4$ are a pair of *Möbius tetrahedra*, each inscribed in the other). Moreover, the five points $S_{1234}, S_{1235}, S_{1245}, S_{1345}, S_{2345}$ all lie on one plane σ_{12345}; the six planes whose subscripts are five of the numbers 1, 2, 3, 4, 5, 6 all pass through one point S_{123456}; and so on. This chain of theorems, due to Homersham Cox, yields a self-dual configuration of 2^{n-1} points and 2^{n-1} planes, with n of the planes through each point and n of the points on each plane.

Instead of defining $S_{\lambda\mu}$ to be an *arbitrary* point on the line $\sigma_\lambda \cdot \sigma_\mu$, we may take it to be the second intersection of that line with a fixed sphere through S. Then each plane yields a circle, and we have a configuration of 2^{n-1} points and 2^{n-1} circles on the sphere or, by stereographic projection, on the inversive plane. The corresponding chain of theorems about circles is due to W.K. Clifford.

By interpreting the subscripts of S or σ as the positions of the positive 1's in the coordinate symbol $(\pm 1, \pm 1, \ldots, \pm 1)$, so that S and σ_1 represent the points $(-1, -1, \ldots, -1)$ and $(1, -1, \ldots, -1)$, we relate the 2^{n-1} points to the vertices of $\mathrm{h}\gamma_n$, and the 2^{n-1} planes (or circles) to the remaining vertices of γ_n. Thus D_n is the group of automorphisms of either configuration, while B_n is the whole group of automorphisms and dualities.

For the infinite group $\tilde{D}_n$ $(n \geqq 4)$, the $n+1$ mirrors of the kaleidoscope may conveniently be taken to have the equations

$$x_1+x_2=0, \qquad\qquad\qquad\qquad x_{n-1}=x_n,$$
$$x_2=x_3, \quad x_3=x_4, \quad \dots, \quad x_{n-2}=x_{n-1},$$
$$x_1=x_2, \qquad\qquad\qquad\qquad x_{n-1}+x_n=1$$

[9b, p. 457]. More precisely, the fundamental region satisfies the inequalities

$$x_1+x_2\geqq 0, \quad x_1\leqq x_2\leqq \dots \leqq x_{n-1}\leqq x_n, \quad x_{n-1}+x_n\leqq 1,$$

and the four *special* vertices (indicated by dots that belong to only one link, and determined by n of the $n+1$ equations) are

$$(\tfrac{1}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}, \tfrac{1}{2}), \quad (0,0,\dots,0,1),$$
$$(-\tfrac{1}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}, \tfrac{1}{2}), \quad (0,0,\dots,0,0)$$

[9b, p. 461]. The first mirror reflects $(x_1, x_2, \dots)$ into $(-x_2, -x_1, \dots)$, while the last reflects $(\dots, x_{n-1}\ x_n)$ into $(\dots, 1-x_n, 1-x_{n-1})$ and the remaining mirrors generate the symmetric group on the n coordinates. Hence $\tilde{D}_n$ is generated by the permutations of the coordinates along with the change of any two different coordinates from (x_μ, x_ν) to $(-x_\mu, -x_\nu)$ or $(x_\mu+1, x_\nu+1)$.

Putting a ring round the last dot (indicating the mirror $x_{n-1}+x_n=1$), we obtain the 'half cubic honeycomb' $\mathrm{h}\delta_{n-1}$, whose typical vertex, given by

$$-x_2=x_1=x_2=\dots=x_n,$$

is the origin. Applying $\tilde{D}_n$, we see that all the vertices are just the points whose n coordinates are integers having an even sum [21, p. 158]. (These points form a *lattice*, since their position vectors belong to an additive group.) By removing one of the three unringed special dots from the graphical symbol for $\mathrm{h}\delta_{n+1}$, we see that the cells of the honeycomb are β_n and $\mathrm{h}\gamma_n$.

The 'quarter cubic honeycomb' $\mathrm{q}\delta_{n+1}$ [19, pp. 47–48] (which is *not* a lattice) is derived from the graph for $\tilde{D}_n$ by putting rings round *two* dots: one at the left end and one at the right end, say the first dot and the last. Accordingly, a typical vertex, given by the equations

$$x_1=x_2=\dots=x_n, \quad x_1+x_2=1-x_{n-1}-x_n,$$

is $(\tfrac{1}{4}, \tfrac{1}{4}, \dots, \tfrac{1}{4}, \tfrac{1}{4})$. The mirrors $x_1+x_2=0$ and $x_{n-1}+x_n=1$ reflect this to

$$(-\tfrac{1}{4}, -\tfrac{1}{4}, \tfrac{1}{4}, \dots, \tfrac{1}{4}, \tfrac{1}{4}) \quad\text{and}\quad (\tfrac{1}{4}, \tfrac{1}{4}, \dots, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{3}{4}).$$

Before applying other operations belonging to the group $\tilde{D}_n$, we may conveniently make a change of coordinates

$$y_\nu=2x_\nu-\tfrac{1}{2} \quad\text{or}\quad x_\nu=\tfrac{1}{2}(y_\nu+\tfrac{1}{2}),$$

so that those three vertices becomes

$$(0,0,\dots,0,0), \quad (-1,-1,0,\dots,0), \quad (0,\dots,0,1,1)$$

and $\tilde{D}_n$ is now generated by the permutations of $y_1, \dots, y_n$ along with the change from (y_μ, y_ν) to $(-y_\mu-1, -y_\nu-1)$ or $(y_\mu+2, y_\nu+2)$ (with $\mu\neq\nu$).

We conclude that the vertices of $q\delta_{n+1}$ can be selected from the totality of points having n integral coordinates (that is, from the vertices of the n-dimensional cubic lattice δ_{n+1}) by the following rule: *if* m_κ *of the coordinates are congruent to* κ (mod 4), *so that* $m_0+m_1+m_2+m_3=n$, *then*

$$m_1 \equiv m_2 \equiv m_3 \pmod 2.$$

To verify this rule, suppose the residues (mod 4) of (y_μ, y_ν) are

$$(0,0),\quad (1,0),\quad (2,0),\quad (3,0),\quad (1,1),\quad (2,1),\quad (3,1),\quad (2,2),\quad (3,2),\quad (3,3).$$

Then subtraction from -1 (or 3) yields

$$(3,3),\quad (2,3),\quad (1,3),\quad (0,3),\quad (2,2),\quad (1,2),\quad (0,2),\quad (1,1),\quad (0,1),\quad (0,0),$$

while adding 2 yields

$$(2,2),\quad (3,2),\quad (0,2),\quad (1,2),\quad (3,3),\quad (0,3),\quad (1,3),\quad (0,0),\quad (1,0),\quad (1,1).$$

In every case, the three numbers m_1, m_2, m_3 are either all changed by 1 or else all changed by 0 or 2.

By removing one of the special dots from the graphical symbol for $q\delta_{n+1}$, we see that the cells of this honeycomb are $h\gamma_n$ and $h_{n-1}\gamma_n$, reducing to β_4 and $t_1\gamma_4$ when $n=4$ (see page 575 of Part II). In fact, the graphical symbol implies $q\delta_5 = t_1\delta_5$, since when $n=4$ the four special dots are no longer separated into two pairs ('left' and 'right') as they are when $n>4$.

Although analogy makes it reasonable to define $D_3 = A_3$, $\tilde{D}_3 = \tilde{A}_3$ and $h\delta_4 = \alpha_3 h$ [11, p. 368], the meaning of $q\delta_{n+1}$ when $n=3$ is not immediately clear. However, the description in terms of coordinates remains valid: it yields the permutations of the 3-dimensional coordinates

$$(0,0,0),\qquad (1,1,0),\qquad (2,2,0),\qquad (3,3,0),\qquad (3,2,1)$$

(mod 4), which are easily recognized as the vertices of the honeycomb $q\delta_4$ of tetrahedra and truncated tetrahedra described on page 402 of Part I, that is, the honeycomb whose hexagonal faces form the regular skew polyhedron $\{6, 6 \mid 3\}$ [19, p. 77].

3.3. The Sporadic Reflection Groups E_n and $\tilde{E}_n$

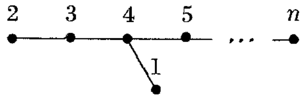
Euclidean spaces of 6, 7 and 8 dimensions admit, along with the 'classical' groups, the 'sporadic' groups

$$[3^{n-4,2,1}] = E_n \qquad (n=6,7,8),$$
$$[3^{2,2,2}] = \tilde{E}_6, \qquad [3^{3,3,1}] = \tilde{E}_7, \qquad [3^{5,2,1}] = \tilde{E}_8.$$

For some purposes it is desirable to augment this family so as to include the infinite group $E_9 = \tilde{E}_8$ and, by gradually cutting off the longest 'tail' of the graph,

$$E_5 = [3^{1,2,1}] = D_5, \qquad E_4 = [3^{0,2,1}] = [3^3] = A_4, \qquad E_3 = A_2 \times A_1. \qquad (3.31)$$

Numbering the generators of E_n in the order



we see that the general presentation

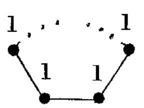
$$(R_\mu R_\nu)^{q_{\mu\nu}} = 1 \qquad (q_{\mu\mu} = 1)$$

[21, p. 188] becomes

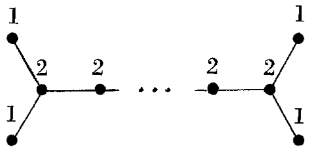
$$R_\nu^2 = (R_1 R_4)^3 = (R_2 R_3)^3 = (R_3 R_4)^3 = \dots = (R_{n-1} R_n)^3 = 1,$$

while all other pairs of generators commute ($q_{\mu\nu} = 2$). To compute the order $72 \times 6!$ of E_6, we can enumerate the 27 cosets of the subgroup $E_5 = D_5$ (of order $2^4 5!$) generated by $R_1, R_2, \dots, R_5$, or the 72 cosets of the symmetric subgroup $\mathfrak{S}_6 = A_5$ generated by $R_2, \dots, R_6$. To compute the order $8 \times 9!$ of E_7, we can enumerate the 56 cosets of the subgroup E_6 generated by R_1, $R_2, \dots, R_6$. Finally, to compute the order $192 \times 10!$ of E_8, we can enumerate the 240 cosets of the subgroup E_7 generated by $R_1, R_2, \dots, R_7$.

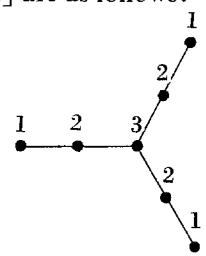
The groups A_n, D_n, E_n, $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_n$ are said to be *trigonal* because in these cases every $q_{\mu\nu} = 2$ or 3, and thus the graph has no marked (or repeated) links. Such an unmarked graph represents an *infinite* (Euclidean) kaleidoscope if and only if 'weights' can be assigned to the dots in such a way that the weight of each dot is equal to half the sum of the weights of its neighbours [21, p. 178 (10·34)]. When this happens we naturally take the smallest weight to be 1; the dots with weight 1 are said to be *special* [6, p. 87; 40, p. 507]. There are (say) f special dots. The actual cases [9a, p. 265; 9b, p. 481] are as follows:



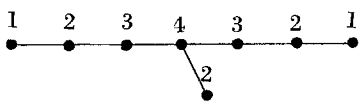
$\tilde{A}_n$ ($f = n+1$)



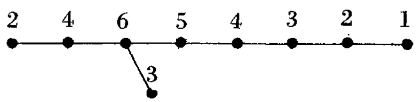
$\tilde{D}_n$ ($f = 4$)



$\tilde{E}_6$ ($f = 3$)



$\tilde{E}_7$ ($f = 2$)



$\tilde{E}_8$ ($f = 1$)

This simple criterion seems mysterious, almost magic. For an explanation, we recall that the fundamental region for any infinite reflection group is a polytope whose dihedral angles are of the form $\pi/q_{\mu\nu}$, where $q_{\mu\nu}$ (with $\mu \neq \nu$) is an integer greater than 1 [21, p. 80]. Such a polytope, in Euclidean n-space, is either a simplex or a Cartesian product of simplexes [14, p. 599], so we may concentrate our attention on a Euclidean n-simplex with dihedral angles $\pi/q_{\mu\nu}$. Let c_ν denote the $(n-1)$-dimensional content of the facet that lies in the νth mirror, expressed as a multiple of the content of the *smallest* facet (so that $c_\nu = 1$ for at least one value of ν and $c_\nu > 1$ for all other values). By filling the νth facet with orthogonal projections of all the others (as in § 1.3 on page 386), we see that

$$c_\nu = \sum_{\mu \neq \nu} c_\mu \cos \pi/q_{\mu\nu}.$$

Since $\cos \pi/2 = 0$, the only non-vanishing terms in this sum are those in which the μth dot is a neighbour of the νth. In the 'trigonal' cases, since $\cos \pi/3 = \frac{1}{2}$, we have

$$c_\nu = \tfrac{1}{2} \sum c_\mu, \tag{3.32}$$

summed over all these neighbours. Thus the contents c_ν, of the facets of the simplex, can be identified with the 'weights' that played such a magically useful role in the criterion for the fundamental region to be a Euclidean simplex. We may equally well describe these numbers c_ν as the magnitudes of vectors with sum zero along outward normals to the mirrors, that is, the magnitudes of 'forces in equilibrium' [48, p. 92; 21, p. 190, where c_ν appears as z^ν; 40, p. 507].

After removing one of the f 'special' facets (of unit content), we are left with n mirrors forming a finite kaleidoscope at the opposite vertex of the simplex. This is one of the f sharpest corners, and the n reflections generate a 'special' subgroup G of the infinite group $\tilde{G}$. For the order Γ of G, Weyl [21, pp. 205–207] discovered the remarkable formula

$$\Gamma = n!\, f \prod c_\nu. \tag{3.33}$$

Writing $|A_n|$ for the order of A_n, and so on, we thus have

$$|A_n| = n!\,(n+1) = (n+1)!,$$
$$|D_n| = n! \times 4 \times 2^{n-3} = 2^{n-1} n!,$$
$$|E_6| = 6! \times 3 \times 2^3 \times 3 = 72 \times 6!,$$
$$|E_7| = 7! \times 2 \times 2^3 \times 3^2 \times 4 = 8 \times 9!,$$
$$|E_8| = 8! \times 2^2 \times 3^2 \times 4^2 \times 5 \times 6 = 192 \times 10!.$$

For the extension of (3.33) to graphs having marked links, see [16, p. 413] and [6, p. 178]. Two more ways to compute the order Γ will be found in § 3.5 and § 3.9.

3.4. Schläfli Functions

As we saw in § 2.2 (on page 563), the fundamental region for the symmetry group $[p, q, r, \ldots]$ of a regular polytope $\{p, q, r, \ldots\}$ in n dimensions is a spherical $(n-1)$-simplex whose bounding hyperplanes have a natural ordering such that any two non-consecutive hyperplanes are perpendicular. The dihedral angles

$\alpha, \beta, \gamma, \ldots$ between pairs of consecutive hyperplanes take the values π/p, π/q, $\pi/r, \ldots$, but it is sometimes useful to consider such an $(n-1)$-dimensional spherical *orthoscheme* for arbitrary values of $\alpha, \beta, \gamma, \ldots$. In particular, we may have $\alpha=\beta=\gamma=\ldots\pi/2$, in which case we speak of an *orthant*: a quadrant when $n=2$, an octant when $n=3$, and so on.

Schläfli [49, p. 260] defined his function

$$f(\alpha, \beta, \gamma, \ldots)$$

to be the $(n-1)$-dimensional content of the spherical orthoscheme divided by the content of the orthant, so that

$$f(\tfrac{1}{2}\pi, \tfrac{1}{2}\pi, \ldots, \tfrac{1}{2}\pi)=1, \qquad f(\alpha, \beta, \gamma, \ldots)=f(\ldots, \gamma, \beta, \alpha),$$
$$f(\alpha)=2\alpha/\pi, \qquad f(\alpha, \beta)=f(\alpha)+f(\beta)-1,$$
$$f(\alpha, \beta, \gamma, \ldots)+f(\pi-\alpha, \beta, \gamma, \ldots)=2f(\beta, \gamma, \ldots). \tag{3.41}$$

In particular,

$$f(\tfrac{1}{2}\pi, \beta, \gamma, \ldots)=f(\beta, \gamma, \ldots).$$

Less obviously [49, p. 255; 50, pp. 173, 246],

$$f(\alpha, \beta, \gamma, \delta)=f(\alpha, \beta, \gamma)+f(a)f(\delta)+f(\beta, \gamma, \delta)-\{f(\alpha)+f(\beta)+f(\gamma)+f(\delta)\}+2,$$

and if $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ [49, pp. 263, 268; 50, pp. 159, 175, 252],

$$2f(\alpha, \beta, \gamma)=\{f(\beta)\}^2-\{1-f(\gamma)\}^2-\{1-f(\alpha)\}^2.$$

In particular, since $f(\pi/4)=\frac{1}{2}$, $f(\pi/3)=\frac{2}{3}$, $f(\pi/5)=\frac{2}{5}$ and $f(2\pi/5)=\frac{4}{5}$,

$$f\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{3}\right)=\frac{1}{24}, \quad f\left(\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{3}\right)=\frac{1}{72},$$

$$f\left(\frac{\pi}{5}, \frac{\pi}{3}, \frac{2\pi}{5}\right)=\frac{1}{45}, \quad f\left(\frac{2\pi}{5}, \frac{\pi}{5}, \frac{\pi}{3}\right)=\frac{1}{225}, \quad f\left(\frac{\pi}{3}, \frac{2\pi}{5}, \frac{\pi}{5}\right)=\frac{19}{225}.$$

Also, since $f\left(\frac{4\pi}{5}, \frac{2\pi}{5}, \frac{\pi}{3}\right)+f\left(\frac{\pi}{5}, \frac{2\pi}{5}, \frac{\pi}{3}\right)=2f\left(\frac{2\pi}{5}, \frac{\pi}{3}\right)=\frac{14}{15}$,

$$f\left(\frac{4\pi}{5}, \frac{2\pi}{5}, \frac{\pi}{3}\right)=\frac{191}{225}.$$

Two special cases of Schläfli's remarkable formula $f_i^m(\alpha)=\binom{i}{m}f_i^0(\alpha)$ [49, p. 267; 50, p. 256] are

$$f\left(2\alpha, \alpha, \frac{\pi}{3}\right)=4f\left(\alpha, \frac{\pi}{3}, \frac{\pi}{3}\right) \quad \text{and} \quad f(\alpha, 2\alpha, \alpha)=6f\left(\alpha, \frac{\pi}{3}, \frac{\pi}{3}\right).$$

Since $f\left(\frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right)+f\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right)=2f\left(\frac{\pi}{3}, \frac{\pi}{3}\right)=\frac{2}{3}$, it follows that

$$f\left(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3}\right)=\frac{2}{15}$$

and since $4f\left(\frac{\pi}{5},\frac{\pi}{3},\frac{\pi}{3}\right)=f\left(\frac{2\pi}{5},\frac{\pi}{5},\frac{\pi}{3}\right)=\frac{1}{225}$,

$$f\left(\frac{\pi}{5},\frac{\pi}{3},\frac{\pi}{3}\right)=\frac{1}{900}.$$

This last result, in the form $F_4(\pi/5)=1/900$, was used in § 2.2 (on page 565) to obtain the order 14400 for [5, 3, 3].

More generally, he defined

$$F_n(\alpha)=f\left(\alpha,\frac{\pi}{3},\frac{\pi}{3},\dots\right)\quad \text{with } n-2 \text{ occurrences of } \pi/3$$

and

$$G_n(\beta)=f\left(\beta,\frac{\pi}{4},\frac{\pi}{3},\dots\right)\quad \text{with } n-3 \text{ occurrences of } \pi/3$$

[50, pp. 177–179]. Thus

$$\begin{aligned} &F_2(\alpha)=f(\alpha)=2\alpha/\pi, && F_3(\alpha)=f(\alpha,\pi/3)=f(\alpha)-\tfrac{1}{3},\\ &G_2(\beta)=f(\beta)=2\beta/\pi, && G_3(\beta)=f(\beta,\pi/4)=f(\beta)-\tfrac{1}{2},\\ &F_4(\pi/3)=\tfrac{2}{15},\quad F_4(\pi/4)=\tfrac{1}{24}, && G_4(\pi/3)=\tfrac{1}{72}. \end{aligned}$$

Since, in spherical $(n-1)$-space, a regular simplex with dihedral angle 2α can be dissected (by 'simplicial subdivision') into $n!$ orthoschemes $f(\alpha, \pi/3, \pi/3, \dots)$, the content of such a regular simplex is

$$n!\,F_n(\alpha)$$

[19, pp. 182–183]. Since the orthant is a regular simplex with dihedral angle $\pi/2$,

$$F_n(\pi/4)=1/n!$$

[49, p. 267; 50, pp. 177–178, 258], in agreement with the natural convention

$$F_0(\alpha)=F_1(\alpha)=1.$$

Since a regular spherical cross polytope with dihedral angle 2β can be dissected into $2^{n-1}(n-1)!$ orthoschemes

$$f(\beta,\pi/4,\pi/3,\pi/3,\dots),$$

the content of such a cross polytope is

$$2^{n-1}(n-1)!\,G_n(\beta).$$

Since a regular Euclidean simplex has dihedral angle $\pi-2\psi=\operatorname{arc\,sec} n$ in n dimensions, or $\operatorname{arc\,sec}(n-1)$ in $n-1$ dimensions [21, p. 295] while a cross polytope has dihedral angle 2 arc sec $\sqrt{n}$ in n dimensions, or 2 arc sec $\sqrt{n-1}$ in $n-1$ dimensions, and since an infinitesimal spherical polytope is Euclidean, we deduce that, if $n \geqq 2$,

$$\begin{aligned} \mathrm{F}_n(\alpha) &= 0 \quad \text{when} \quad \alpha = \tfrac{1}{2} \operatorname{arc\,sec}(n-1) \\ \mathrm{G}_n(\beta) &= 0 \quad \text{when} \quad \beta = \operatorname{arc\,sec} \sqrt{n-1}. \end{aligned} \tag{3.42}$$

Schläfli derived most of the above results from his amazing discovery [49, p. 235; 50, pp. 167, 234] that if V is the content of any spherical $(n-1)$-simplex, expressed as a function of all its $\binom{n}{2}$ dihedral angles λ, then

$$(n-2)\, dV = \sum l\, d\lambda,$$

where l is the content of the $(n-3)$-face at which the dihedral angle λ takes place. The case $n=3$ agrees with Girard's formula for the area of the spherical triangle, if we make the convention that the 'content' of a single point is 1. The case $n=4$ was elucidated by H.W. Richmond [18, p. 286].

Since the ratio of contents of the $(n-1)$-sphere and $(n-3)$-sphere is $2\pi/(n-2)$ [21, p. 126], the ratio of the corresponding orthants is $\pi/2(n-2)$. Accordingly, when V and l are expressed in terms of orthants, the coefficient $(n-2)$ in Schläfli's formula has to be changed to $(\pi/2)$, and thus

$$dV = \frac{2}{\pi} \sum l\, d\lambda = \sum l\, df(\lambda).$$

As two special cases, we have

$$d\mathrm{F}_n(\alpha) = \frac{2}{\pi} \mathrm{F}_{n-2}(a)\, d\alpha,$$

where a is given in terms of α by

$$\cos a = \frac{\sin \alpha}{\sqrt{4 \sin^2 \alpha - 1}} \quad \text{or} \quad \sec 2a = \sec 2\alpha - 2 \tag{3.431}$$

[50, p. 256; 19, p. 183], and

$$d\mathrm{G}_n(\beta) = \frac{2}{\pi} \mathrm{F}_{n-2}(b)\, d\beta,$$

where b is given in terms of β by

$$\cos b = \frac{\sin \beta}{\sqrt{4 \sin^2 \beta - 2}} \quad \text{or} \quad \sec 2b = \sec^2 \beta - 2 \tag{3.432}$$

[50, p. 259]. In view of (3.42), we thus have, for any $\phi \geqq \frac{1}{2}$ arc sec$(n-1)$,

$$\mathrm{F}_n(\phi) = \frac{2}{\pi} \int_{\frac{1}{2} \operatorname{arc\,sec}(n-1)}^{\phi} \mathrm{F}_{n-2}(a)\, d\alpha$$

and for any $\phi \geqq \operatorname{arc\,sec}\sqrt{n-1}$,

$$G_n(\phi)=\frac{2}{\pi}\int_{\operatorname{arc\,sec}\sqrt{n-1}}^{\phi} F_{n-2}(b)\,d\beta.$$

To facilitate computation [19, pp. 184, 194] let us introduce new variables

$$u=\sec 2\alpha=2+\sec 2a, \qquad v=\sec^2\beta=2+\sec 2b$$

and new functions

$$\begin{aligned} f_n(u)&=f_n(\sec 2\alpha)=F_n(\alpha),\\ g_n(v)&=g_n(\sec^2\beta)=G_n(\beta). \end{aligned} \tag{3.44}$$

Then

$$du=d(\sec 2\alpha)=2\sec 2\alpha\tan 2\alpha\,d\alpha=2u\sqrt{u^2-1}\,d\alpha,$$

and

$$dv=d(\sec^2\beta)=2\sec^2\beta\tan\beta\,d\beta=2v\sqrt{v-1}\,d\beta.$$

Since

$$F_{n-2}(a)=f_{n-2}(\sec 2a)=f_{n-2}(u-2)$$

and

$$F_{n-2}(b)=f_{n-2}(\sec 2b)=f_{n-2}(v-2),$$

we have

$$F_{n-2}(a)\,d\alpha=f_{n-2}(u-2)\frac{du}{2u\sqrt{u^2-1}}$$

and

$$F_{n-2}(b)\,d\beta=f_{n-2}(v-2)\frac{dv}{2v\sqrt{v-1}},$$

enabling us to define (for $x\geqq n-1$) our new 'Schläfli functions' $f_n(x)$ and $g_n(x)$ recursively by

$$\boxed{\begin{aligned} &f_0(x)=f_1(x)=1,\\ &f_n(x)=\frac{1}{\pi}\int_{n-1}^{x}\frac{f_{n-2}(u-2)\,du}{u\sqrt{u^2-1}} \qquad (n\geqq 2),\\ &g_n(x)=\frac{1}{\pi}\int_{n-1}^{x}\frac{f_{n-2}(u-2)\,du}{u\sqrt{u-1}} \qquad (n\geqq 2). \end{aligned}} \tag{3.45}$$

In particular,

$$\begin{aligned} f_2(x)&=\frac{1}{\pi}\int_1^x\frac{du}{u\sqrt{u^2-1}}=\frac{1}{\pi}\operatorname{arc\,sec}x,\\ g_2(x)&=\frac{1}{\pi}\int_1^x\frac{du}{u\sqrt{u-1}}=\frac{2}{\pi}\operatorname{arc\,sec}\sqrt{x}. \end{aligned} \tag{3.46}$$

Schläfli [50, pp. 169, 178–179] observed that, when n is odd, $F_n(\phi)$ and $G_n(\phi)$ can be expressed in terms of the same functions with smaller values of n, and so ultimately with smaller *even* values. In terms of $f_n(x)$ and $g_n(x)$, his formulae become

$$\begin{aligned} f_{2m+1}(x) &= \sum_{\nu=0}^{m} (-1)^\nu a_{2\nu+1} f_{2m-2\nu}(x), \\ g_{2m+1}(x) &= \sum_{\nu=0}^{m-1} (-1)^\nu a_{2\nu+1} g_{2m-2\nu}(x) + (-1)^m a_{2m}, \end{aligned} \tag{3.47}$$

where a_n is defined recursively by $a_0 = a_1 = 1$ and, for $n \geqq 2$,

$$2n a_n = \sum_{\nu=0}^{n-1} a_\nu a_{n-1-\nu}.$$

It follows [44, p. 105] that a_n is the coefficient of θ^n in the two Maclaurin series

$$\begin{aligned} \sec\theta &= 1 + \tfrac{1}{2}\theta^2 + \tfrac{5}{24}\theta^4 + \tfrac{61}{720}\theta^6 + \tfrac{277}{8064}\theta^8 + \dots, \\ \tan\theta &= \theta + \tfrac{1}{3}\theta^3 + \tfrac{2}{15}\theta^5 + \tfrac{17}{315}\theta^7 + \tfrac{62}{2835}\theta^9 + \dots \end{aligned}$$

[43, pp. 328–329; 1, pp. 387–388].

It will be found useful to observe also that, by (3.42) and (3.44), or directly from (3.45),

$$f_{n+1}(n) = g_{n+1}(n) = 0;$$

therefore, when n is even,

$$\begin{aligned} f_n(n) &= a_3 f_{n-2}(n) - a_5 f_{n-4}(n) + \dots \mp a_{n-1} f_2(n) \pm a_{n+1}, \\ g_n(n) &= a_3 g_{n-2}(n) - a_5 g_{n-4}(n) + \dots \mp a_{n-1} g_2(n) \pm a_n, \end{aligned} \tag{3.48}$$

and (3.45) may be expressed in John Leech's computationally convenient form

$$\begin{aligned} f_n(x) &= f_n(n) + \frac{1}{\pi}\int_n^x \frac{f_{n-2}(u-2)\,du}{u\sqrt{u^2-1}}, \\ g_n(x) &= g_n(n) + \frac{1}{\pi}\int_n^x \frac{f_{n-2}(u-2)\,du}{u\sqrt{u-1}} \end{aligned} \tag{3.49}$$

[19, p. 194]. In particular, (3.48) and (3.46) yield

$$\begin{aligned} f_4(4) &= \frac{1}{3} f_2(4) - \frac{2}{15} = \frac{1}{3}\left(\frac{\operatorname{arc\,sec} 4}{\pi} - \frac{2}{5}\right), \\ g_4(4) &= \frac{1}{3} g_2(4) - \frac{5}{24} = \frac{1}{3}\left(\frac{2}{3} - \frac{5}{8}\right) = \frac{1}{72}. \end{aligned}$$

3.5. A New Formula for the Order of E_n

In this section we will see that the functions (3.45), which measure the contents of regular simplexes and cross-polytopes in spherical $(n-1)$-space, can be combined to yield the content of the fundamental region for the 'sporadic' group

$$E_n = [3^{n-4,2,1}].$$

When the generating reflections for this group are numbered as on page 10 we shall find it convenient to let r_ν denote the mirror for R_ν, so that the fundamental region is a simplex having dihedral angles $\pi/3$ between the pairs of facets

$$r_1 r_4, \quad r_2 r_3, \quad r_3 r_4, \quad r_4 r_5, \quad \ldots, \quad r_{n-1} r_n$$

and right angles everywhere else. As in (3.31), we let this family of groups include not only $E_9 = \tilde{E}_8$ but also

$$E_5 = D_5, \quad E_4 = A_4, \quad E_3 = A_2 \times A_1.$$

In the last case the graph is no longer connected and the fundamental region is an isosceles spherical triangle DAB with angles $\pi/3$, $\pi/2$, $\pi/2$ opposite to its sides $r_1 = AB$, $r_2 = BD$, $r_3 = DA$, which are the mirrors for R_1, R_2, R_3, as in Fig. 3.5a.

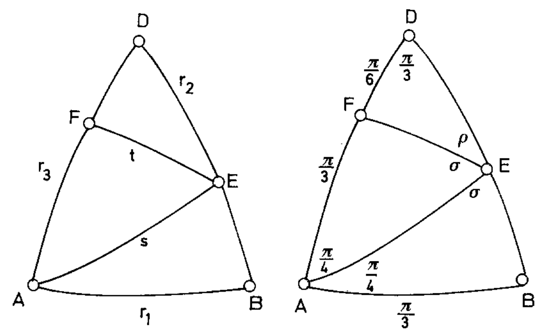


Fig. 3.5a. $(E_3) = F_3(\rho) + 2G_3(\sigma)$

Analogy with Fig. 6.7A of *Regular Polytopes* [21, p. 111] suggests a dissection of this spherical triangle into 3 smaller right-angled triangles by cutting along the bisector $s = AE$ of the right angle at A and along the arc $t = EF$ perpendicular to AD. Since s serves as a mirror reflecting ABE into AFE, these two triangles have equal angles σ at E. Let ρ denote the remaining angle at E, that is, $\rho = \angle DEF$. This dissection shows that the area of triangle DAB is

$$(E_3) = F_3(\rho) + 2G_3(\sigma),$$

where ρ and σ can be evaluated as follows. The congruent triangles ABE and AFE yield also $AF = AB = \pi/3$. Since $AD = \pi/2$, $DF = AD - AF = \pi/6$. Since the angle D is $\pi/3$, the classical formula $\cos A = \cos a \sin B$ (for a right-angled spherical triangle ABC) yields

$$\cos\rho = \cos\frac{\pi}{6}\sin\frac{\pi}{3} = \frac{3}{4}, \qquad \cos\sigma = \cos\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{1}{2\sqrt{2}}, \tag{3.51}$$

whence, as we see at once from Fig. 3.5a,

$$\rho + 2\sigma = \pi. \tag{3.52}$$

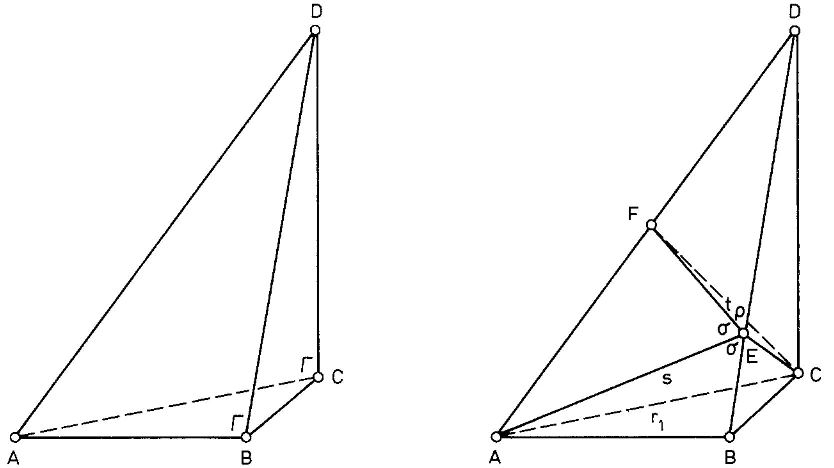


Fig. 3.5b. $(E_4)=F_4(\rho)+2G_4(\sigma)$

In an orthoscheme *ABCD*, with right angles *ABC*, *ABD*, *ACD*, *BCD* in its four faces, draw *CE* perpendicular to *BD*, and *EF* perpendicular to *AD*, as in Figure 3.5b. Consider the two skew lines *AD* and *CE*. *AD* lies in the plane *ABD*, which is perpendicular to *CE* (since the dihedral angle along *BD* is a right angle), so *CE* must lie in a plane perpendicular to *AD*, namely the plane *CEF* [36]. Thus *AD* is perpendicular to *CF*. In other words, all the four angles at *F* (in the planes *ABD* and *ACD*) are right angles. It follows that the two planes *ACE* and *CEF* serve to decompose the orthoscheme *ABCD* into three smaller orthoschemes *ABEC*, *AFEC*, *DFEC*, with a common edge *EC*.

This decomposition of an orthoscheme belongs to 'absolute' geometry, and thus holds, in particular, for the spherical fundamental region of the reflection group $E_4=A_4$. Now the dihedral angles along *AB*, *AD*, *CD* are all $\pi/3$. To facilitate extending these ideas from E_4 to E_n, let us name the four mirrors in the unusual order

$$r_1=ABC, \quad r_2=BCD, \quad r_3=ACD, \quad r_4=ABD,$$

and call the cutting planes $s=ACE$, $t=CEF$. The equal dihedral angles along *AB* and *AD* provide an isosceles 'trihedral angle' at *A*, so the two orthoschemes *ABEC* and *AFEC* are (oppositely) congruent by reflection in the plane $s=ACE$, which now bisects the right dihedral angle along *AC*. In fact, s could have been defined as the bisector of that dihedral angle, and t as the image of r_2 by reflection in s. Naming faces as well as vertices, we may now say that the two planes s and t decompose the orthoscheme

$$ABCD=r_2\, r_3\, r_4\, r_1$$

into the three smaller orthoschemes

$$ABEC=r_2\, s\, r_1\, r_4, \quad AFEC=t\, s\, r_3\, r_4, \quad DFEC=t\, r_2\, r_3\, r_4.$$

On a sphere with centre C, the lines and planes through C cut out an arrangement of points and arcs just like Fig. 3.5a. Thus the dihedral angles along the common edge CE of the three small orthoschemes are σ, σ, ρ. Since CE is perpendicular to the face ABD of the large orthoscheme, these same angles occur at E in that face. Also there is a right angle at B; but the angle arc sec 3 $\approx 70° 32'$ at A is less than a right angle, and the angle $\frac{1}{2}$ arc sec$(-3) \approx 54° 44'$ at D is less than $\pi/3$. It follows that the volume of $ABEC$ or $AFEC$, in terms of the orthant as unit, is

$$f(\sigma, \pi/4, \pi/3) = \mathrm{G}_4(\sigma)$$

and the volume of $DFEC$ is

$$f(\rho, \pi/3, \pi/3) = \mathrm{F}_4(\rho).$$

We have thus succeeded in expressing the volume of the fundamental region for E_4 in the form

$$(E_4) = \mathrm{F}_4(\rho) + 2\mathrm{G}_4(\sigma). \tag{3.53}$$

The extension to n dimensions is now clear. The fundamental region for E_n $(n<9)$ is a spherical $(n-1)$-simplex bounded by the mirrors $r_1, \ldots, r_n$ for $\mathrm{R}_1, \ldots, \mathrm{R}_n$; r_1 is orthogonal to all the other mirrors except r_4. The $n-1$ mirrors $r_1, \ldots, r_{n-1}$ cut out, on the $(n-2)$-sphere centred at their common point, an $(n-2)$-simplex which is the fundamental region for the E_{n-1} generated by $\mathrm{R}_1, \ldots, \mathrm{R}_{n-1}$. This holds also when $n=9$, except that then the 8-simplex is not spherical but Euclidean.

The $(n-1)$-dimensional content

$$(E_n) = \mathrm{F}_n(\rho) + 2\mathrm{G}_n(\sigma), \tag{3.54}$$

for each $n<9$, can be verified by dissecting the simplex into three orthoschemes by means of two suitably placed hyperplanes s and t. The hyperplane s (perpendicular to r_ν for $\nu \geqq 4$) bisects the right dihedral angle between r_1 and r_3, and reflects r_2 into t. Like s, t is perpendicular to r_ν for every $\nu \geqq 4$; t is perpendicular also to r_3, since r_2 is perpendicular to r_1. Thus the two hyperplanes s and t can be used to dissect the original simplex $r_1 r_2 \ldots r_n$ into three orthoschemes

$$r_2 s r_1 r_4 r_5 \ldots r_n, \qquad t s r_3 r_4 r_5 \ldots r_n, \qquad t r_2 r_3 r_4 r_5 \ldots r_n.$$

Since the angles that s and t make with r_1, r_2, r_3 are independent of n, the contents of these orthoschemes are $\mathrm{G}_n(\sigma)$, $\mathrm{G}_n(\sigma)$, $\mathrm{F}_n(\rho)$.

By (3.51), $\cos 2\rho = \frac{1}{8}$ and $\cos^2 \sigma = \frac{1}{8}$. Hence, in the notation of (3.44), we have, for each $n<9$, the simple formula

$$\boxed{(E_n) = \mathrm{f}_n(8) + 2\mathrm{g}_n(8).} \tag{3.55}$$

Since the whole $(n-1)$-sphere is covered by 2^n orthants, it follows that the order $|E_n|$ of the whole group $E_n = [3^{n-4,2,1}]$ is

$$|E_n| = 2^n/(E_n) \qquad (n \leqq 9). \tag{3.56}$$

Since $f_n(n-1)=g_n(n-1)=0$, $|E_n|$ is infinite when $n=9$, that is, for $E_9=\tilde{E}_8$. When n is odd, we can use (3.47) to obtain

$$(E_{2m+1})=\sum_{v=0}^{m-1}(-1)^v a_{2v+1}(E_{2m-2v})+(-1)^m(a_{2m+1}+2a_{2m}). \tag{3.57}$$

By (3.46) and (3.52),

$$(E_2)=f_2(8)+2g_2(8)=\frac{2\rho}{\pi}+2\frac{2\sigma}{\pi}=\frac{2}{\pi}(\rho+2\sigma)=2.$$

This suggests that we should regard E_2 as the group A_1 of order $2^2/2=2$, generated by the single reflection R_2.

By (3.57) with $m=1$,

$$(E_3)=(E_2)-(a_3+2a_2)=2-(\tfrac{1}{3}+1)=\tfrac{2}{3},$$

in agreement with the known order $2^3/\frac{2}{3}=12$ for $E_3=A_1\times A_2\cong\mathfrak{S}_2\times\mathfrak{S}_3$.

When $n=4$ or 6, the integration in 3.49 needs a computer. This difficult task was kindly undertaken by N.J.A. Sloane. Tabulating

$$\frac{\operatorname{arc\,sec}(u-2)}{\pi^2 u\sqrt{u^2-1}} \quad\text{and}\quad \frac{\operatorname{arc\,sec}(u-2)}{\pi^2 u\sqrt{u-1}}$$

for various values of u from 3.5 to 8.2, he used (3.49) with $n=4$ to compute

$$f_4(x)=f_4(4)+\frac{1}{\pi^2}\int_4^x\frac{\operatorname{arc\,sec}(u-2)\,du}{u\sqrt{u^2-1}}$$

for various values of x, including

$$f_4(8)\approx 0.02268\,05970\,96406\,8,$$

and to compute

$$g_4(8)=g_4(4)+\frac{1}{\pi^2}\int_4^8\frac{\operatorname{arc\,sec}(u-2)\,du}{u\sqrt{u-1}}\approx 0.05532\,63681\,18463\,3,$$

whence, by (3.54),

$$(E_4)=0.13333\,33333\,33333\ldots=\tfrac{2}{15},$$

in agreement with the known order $2^4/\frac{2}{15}=120$ for $E_4=A_4\cong\mathfrak{S}_5$. Then (3.57) with $m=2$ yields

$$(E_5)=(E_4)-\tfrac{1}{3}(E_2)+\tfrac{2}{15}+\tfrac{5}{12}=\tfrac{2}{15}-\tfrac{2}{3}+\tfrac{11}{20}=\tfrac{1}{60},$$

in agreement with the known order $2^5 \times 60$ for $E_5 = D_5$. He used (3.48) to compute $f_6(6)$ and $g_6(6)$. Then his tables for $f_4(x)$ enabled him to obtain

$$f_6(8) = f_6(6) + \frac{1}{\pi} \int_6^8 \frac{f_4(u-2)\, du}{u\sqrt{u^2-1}} \approx 0.00018\,73637\,62773\,664,$$

$$g_6(8) = g_6(6) + \frac{1}{\pi} \int_6^8 \frac{f_4(u-2)\, du}{u\sqrt{u-1}} \approx 0.00052\,36020\,69229\,426,$$

whence

$$(E_6) \approx 0.00123\,45679\,0123 \approx 1/810.00000\,000.$$

But we see from the graph that E_6 has a subgroup E_5, whose order is divisible by 2^6; therefore

$$(E_6) = 1/810$$

exactly, and

$$|E_6| = 2^6 \times 810 = 51840,$$

in agreement with page 11.

By (3.57) with $m=3$,

$$(E_7) = (E_6) - \tfrac{1}{3}(E_4) + \tfrac{2}{15}(E_2) - (\tfrac{17}{315} + \tfrac{61}{360})$$
$$= \tfrac{1}{810} - \tfrac{2}{45} + \tfrac{4}{15} - \tfrac{17}{315} - \tfrac{61}{360} = \tfrac{1}{22680},$$

and

$$|E_7| = 2^7 \times 22680 = 2903040.$$

Finally, since $(E_9) = 0$,

$$(E_8) = \tfrac{1}{3}(E_6) - \tfrac{2}{15}(E_4) + \tfrac{17}{315}(E_2) - (\tfrac{62}{2835} + \tfrac{277}{4032})$$
$$= \tfrac{1}{2430} - \tfrac{4}{225} + \tfrac{34}{315} - \tfrac{62}{2835} - \tfrac{277}{4032} = \tfrac{1}{2721600},$$

and

$$|E_8| = 2^8 \times 2721600 = 696729600.$$

3.6. The Groups E_6, $\tilde{E}_6$ and the Lattice 2_{22}

The lists of reflection groups in § 3.2 and § 3.3 include the instances

$$D_{p+3} = [3^{p,1,1}], \quad E_{p+4} = [3^{p,2,1}], \quad \tilde{E}_6 = [3^{2,2,2}] \text{ and } \tilde{E}_7 = [3^{3,3,1}]$$

of the general 'Coxeter group' $[3^{p,q,r}]$ or, in J.H. Conway's very natural notation, Y_{pqr} (because its graph is a triquetra having legs of lengths p, q, r). In the notation of *Regular Polytopes* [21, p. 201], a ring round the foot of the first leg yields the uniform polytope (or honeycomb) p_{qr} $(= p_{rq})$. Removing the other two feet in turn, we see that p_{qr} has facets (or cells) of two types, $p_{(q-1)r}$ and

$p_{q(r-1)}$; their centres are the vertices of q_{rp} and r_{pq}, respectively. Thus the three polytopes

$$p_{qr}, \quad q_{rp}, \quad r_{pq}$$

are related by a kind of *triality* [23, p. 134], analogous to the duality that relates two reciprocal *regular* polytopes. For instance, when $p=q=r=1$, so that we are considering the 16-cell $\beta_4=1_{11}$, alternate tetrahedral facets 1_{01} and 1_{10} have as centres the 8+8 vertices of two other β_4's. In other words, triality relates the three β_4's which can be inscribed in the 24-cell {3, 4, 3} [21, p. 149].

We see from § 2.4 (on page 572) that the vertex figure of p_{qr} is $(p-1)_{qr}$ and thus the 'pth vertex figure' is $0_{qr}=\left\{\begin{matrix}3^q\\3^r\end{matrix}\right\}=\mathrm{t}_r\,\alpha_n$ where $n=q+r+1$. Since the vertex figure of 0_{qr} is the Cartesian product $\alpha_q\times\alpha_r$ of two regular simplexes (written as $[\alpha_q, \alpha_r]$ in [11, p. 359]), we could have defined p_{qr} to be the uniform polytope whose $(p+1)$th vertex figure is $\alpha_q\times\alpha_r$ while its 2-faces are triangles [23, pp. 131–134]. This is actually how the symbol p_{qr} first arose [11, pp. 371–372], three years before the derivation from the reflection group $[3^{p,q,r}]$ was thought of. In particular,

$$p_{q0}=\alpha_{p+q+1}, \quad p_{11}=\beta_{p+3}, \quad 1_{q1}=\mathrm{h}\gamma_{q+3}.$$

From the list of group orders at the end of § 3.3, we can easily compute the number of vertices for each of the remaining polytopes p_{qr}:

for	2_{21},	$\lvert E_6\rvert/\lvert D_5\rvert=27$;	for	1_{22},	$\lvert E_6\rvert/\lvert A_5\rvert=72$;
for	3_{21},	$\lvert E_7\rvert/\lvert E_6\rvert=56$;	for	2_{31},	$\lvert E_7\rvert/\lvert D_6\rvert=126$;
for	1_{32},	$\lvert E_7\rvert/\lvert A_6\rvert=576$;	for	4_{21},	$\lvert E_8\rvert/\lvert E_7\rvert=240$;
for	2_{41},	$\lvert E_8\rvert/\lvert D_7\rvert=2160$;	for	1_{42},	$\lvert E_8\rvert/\lvert A_7\rvert=17280$.

We saw, in § 2.6, that the vertices of $0_{[n]}=\alpha_{n-1}h$ form a *lattice*. More generally [21, p. 205], a lattice arises when we put a ring round any *special* vertex of a graphical symbol, and consequently the vertex figure of such a honeycomb is centrally symmetrical. Thus the vertices of the honeycombs

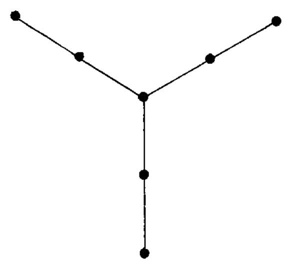
$$0_{[n]}, \quad \mathrm{h}\delta_n, \quad 2_{22}, \quad 3_{31}, \quad 5_{21}$$

form lattices, and their vertex figures

$$\mathrm{e}\alpha_n, \quad \mathrm{t}_1\beta_n, \quad 1_{22}, \quad 2_{31}, \quad 4_{21}$$

are centrally symmetrical. On the other hand, a honeycomb may have a centrally symmetrically vertex figure without forming a lattice; familiar instances are $\left\{\begin{matrix}6\\3\end{matrix}\right\}$ and {3, 4, 3, 3} [21, pp. 64, 158; see also § 2.6 (p. 580)]. Like {3, 6} and {3, 3, 4, 3}, the lattice honeycomb 2_{22} has all its cells congruent, although we may appropriately name them 2_{12} and 2_{21} alternately.

This honeycomb 2_{22} is obtained by ringing one of the three 'feet' of the symmetrical triquetra



The 7 mirrors of the 6-dimensional kaleidoscope $\tilde{E}_6$ may conveniently be taken to have the equations

$$x_1 - x_2 = 0, \qquad x_1 + x_2 + x_3 + x_4 - x_5 + \sqrt{3}\,x_6 = 0,$$
$$x_2 - x_3 = 0, \qquad x_4 - x_5 = 0,$$
$$x_3 - x_4 = 0,$$
$$x_4 + x_5 = 0,$$
$$x_1 + x_2 + x_3 + x_4 + x_5 - \sqrt{3}\,x_6 = 4. \tag{3.61}$$

Using 6 of the 7 equations, we see that the 3 special vertices of this 6-dimensional simplex are

$$(2, 0, 0, 0, 0, -2/\sqrt{3}), \qquad (0, 0, 0, 0, 0, -4/\sqrt{3}),$$
$$(0, 0, 0, 0, 0, 0).$$

Taking the origin to be one vertex of 2_{22}, we reflect in the mirror (3.61) to obtain another: $(1, 1, 1, 1, 1, -\sqrt{3})$. The remaining reflections then yield the lattice consisting of all the points

$$(x_1, x_2, x_3, x_4, x_5\,; \sqrt{3}\,y)$$

where x_ν and y are *six integers, mutually congruent* mod 2, *with sum congruent to* 0 mod 4. These coordinates for 2_{22} were discovered by John Leech [39, p. 154], using a different procedure.

Among the points of this lattice, the 72 nearest to the origin are the vertices of the vertex figure 1_{22}, namely *the* 40 *permutations of* $(\pm 2, \pm 2, 0, 0, 0; 0)$ (keeping x_6 fixed) *and the* 32 *points*

$$(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1; \pm\sqrt{3})$$

with an odd number of minus signs (including the sign of x_6).

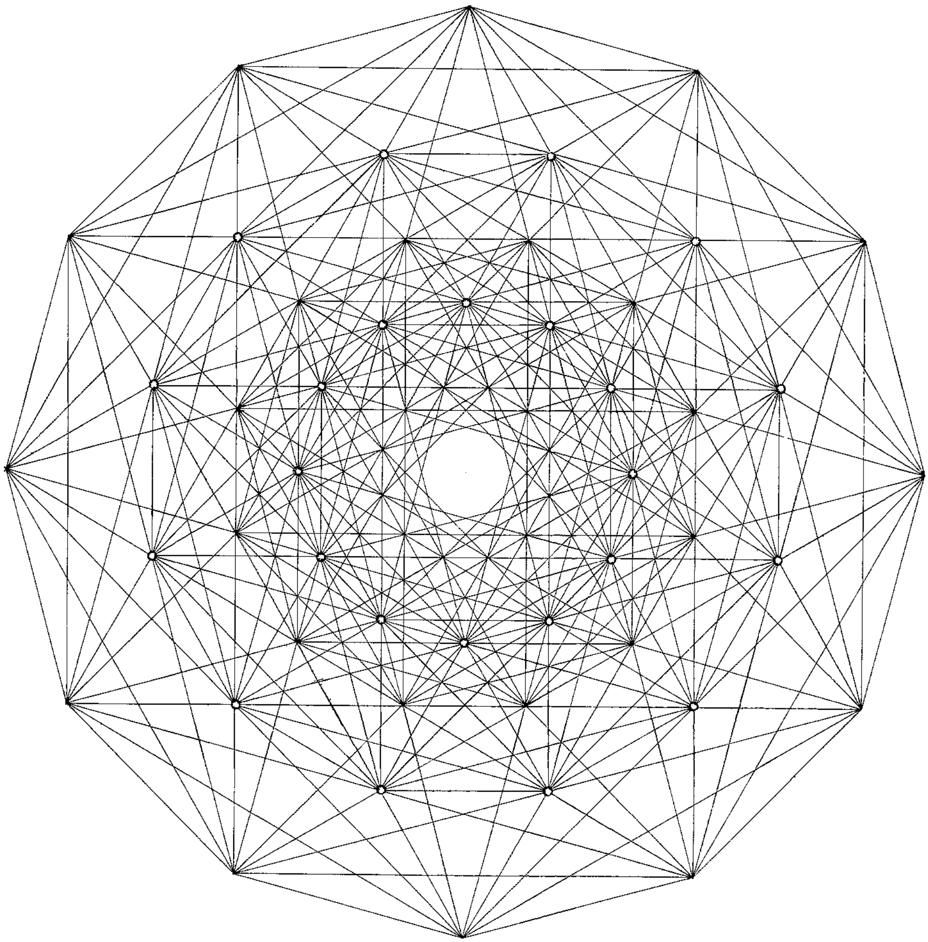


Fig. 3.6a. The six-dimensional polytope 1_{22}

This 6-dimensional polytope 1_{22}, being the vertex figure of the lattice 2_{22}, is centrally symmetrical. Its 36 diameters, joining pairs of opposite vertices, are perpendicularly bisected by hyperplanes which are the mirrors for the 36 reflections belonging to the group E_6 [14, pp. 611–612]. For instance, the locus of points equidistant from $(1, 1, 1, 1, -1, \sqrt{3})$ and $(-1, -1, -1, -1, 1, -\sqrt{3})$ is the mirror

$$x_1+x_2+x_3+x_4-x_5+\sqrt{3}x_6=0.$$

In Peter McMullen's Fig. 3.6a, the 72 vertices appear in sets of 12 on *four* concentric circles: four, not six, because each of the 24 'white' points is really double (one vertex hidden behind another).

Again, among the points of the lattice 2_{22}, the 27 nearest to $(0, 0, 0, 0, 0, 4/\sqrt{3})$ are the vertices of a cell 2_{21}. They consist of *the origin*

$$(0, 0, 0, 0, 0, 0),$$

the 10 *permutations of* $(\pm 2, 0, 0, 0, 0; 2\sqrt{3})$, and *the* 16 *points*

$$(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1; \sqrt{3}) \tag{3.62}$$

with an odd number of minus signs.

Of course, we could just as well use an *even* number of minus signs in (3.62), and thus agree with Schoute [51, p. 376] except that he subtracted $4/\sqrt{3}$ from x_6 so as to put the origin at the centre.

Among the 27 vertices of this remarkable six-dimensional polytope 2_{21} [33, p. 47; 31, p. 660; 15], the 16 given by (3.62) belong to an $h\gamma_5 = 1_{21}$ which is the vertex figure (at the origin) of the 2_{21} and is also one of the 27 facets of 1_{22} of type 1_{21}. The remaining 27 facets $h\gamma_5$ of 1_{22}, namely those of type 1_{12}, can be derived by applying the central inversion which reverses the signs of all the coordinates.

The facets of 2_{21} are of two kinds: 27 cross polytopes $\beta_5 = 2_{11}$ such as $(\pm 2, 0, 0, 0, 0; 2\sqrt{3})$, and 72 simplexes $\alpha_5 = 2_{20}$ such as one joining (1, 1, 1, 1, 1; $\sqrt{3}$) to the permutations of (2, 0, 0, 0, 0; $2\sqrt{3}$).

It was observed by Schoute [51, p. 377], that the 27 vertices of 2_{21} form a *two-distance set*: every two of them belong either to an edge or to a diagonal, and every diagonal is $\sqrt{2}$ times as long as an edge. This observation enabled him to recognize that E_6, the symmetry group of 2_{21}, the isomorphic to the group of automorphisms of the 27 lines on the cubic surface [55; 15; 35, 111–119]. The edges and diagonals correspond to the pairs of skew lines and intersecting lines, respectively. Moreover, the 36 pairs of opposite α_5's correpond to the 36 double-sixes [37, § 25, Fig. 181; 24, p. 118], and the 45 'equatorial' triangles such as

$$(0, 0, 0, 0, 0, 0), \qquad (0, 0, 0, 0; \pm 2; 2\sqrt{3})$$

correspond to the 45 tritangent planes. Completely orthogonal to the plane of such a triangle, there is a 4-flat onto which the whole polytope can be orthogonally projected. In this projection, the triangle is foreshortened into a central point while the remaining 24 vertices of the polytope project into the 24 vertices of a regular 24-cell {3, 4, 3} [56].

The reflection group E_6 has an 'even' subgroup E_6^+, of order $36 \times 6! = 25920$, generated by rotations. This is the simple group

$$U_4(2) \cong O_6^-(2) \cong SU_4(2) \cong S_4(3) \cong O_5(3) \cong C_2(3)$$

[9, p. 26].

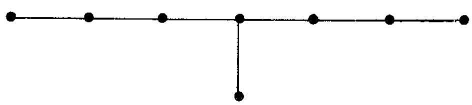
Since the cubic surface with its 27 real lines is a non-orientable surface having Euler-Poincaré characteristic $\chi = -5$ [35, pp. 113–114], it can be unfolded into a 14-gon by cutting along $2 - \chi = 7$ circuits from one point. It was observed by Wolf Barth and Horst Knörrer [3, pp. 10–14] that these 7 circuits (or the 7 generators of the first homology group) correspond to 7 passages that can be seen in a model of the cubic surface. They are associated with 7 of the 36 double-sixes, namely

$$\begin{array}{ccccc} N_{12} & & & & N_{56} \\ & N_{23} & & N_{45} & \\ & & N_{34} & & \\ & & N_{123} & & \\ & & N & & \end{array}$$

in the notation of [35, p. 112]; in other words, they correspond to the 7 dots in the triquetra for $\tilde{E}_6$ (see page 23).

3.7. The Groups E_7, $\tilde{E}_7$ and the Lattice 3_{31}

The infinite group $\tilde{E}_7=[3^{3,3,1}]$ arises from the 7-dimensional kaleidoscope



whose 8 mirrors may be constructed in Euclidean 8-space as sections of the 8 hyperplanes

$$u_1=u_2,\quad u_2=u_3,\quad u_3=u_4,\quad u_4=u_5,\quad u_5=u_6,\quad u_6=u_7,\quad u_7=u_8+2,$$
$$u_1+u_2+u_3+u_4=u_5+u_6+u_7+u_8 \tag{3.71}$$

by the hyperplane $\sum u_\nu=0$. Thus the two *special* vertices of this 7-dimensional simplex are

$$\tfrac{1}{2}(-3;1,1,1,1,1,1;-3) \quad \text{and} \quad (0,0,0,0,0,0,0,0).$$

Reflections in the first six mirrors are simply the transpositions

$$(1\,2),\quad (2\,3),\quad (3\,4),\quad (4\,5),\quad (5\,6),\quad (6\,7)$$

which generate the symmetric group on the first seven coordinates. The seventh mirror, $u_7=u_8+2$, reflects the origin into

$$(0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 2,\ -2).$$

The eighth reflection, naturally denoted by

$$[1234\cdot 5678], \tag{3.72}$$

transforms $(0,\ 0,\ 0,\ 2,\ 0,\ 0,\ 0,\ -2)$ into $(-1,\ -1,\ -1,\ 1,\ 1,\ 1,\ 1,\ -1)$. In fact, the midpoint between these two points lies on the mirror (3.71), while the vector

$$(0,0,0,2,0,0,0,-2)-(-1,-1,-1,1,1,1,1,-1)=(1,1,1,1,-1,-1,-1,-1)$$

is perpendicular to the mirror. A 'bifid' reflection $[efgh\cdot ijkl]$, such as (3.72), is evidently unchanged by any permutation of the four numbers $efgh$ or $ijkl$

and by bodily interchanging these tetrads. The infinite reflection group $\tilde{E}_7$ actually includes the whole symmetric group $\mathfrak{S}_8$, since

$$(7\ 8)=[1237\cdot 4568][4567\cdot 1238][1237\cdot 4568]$$

[11, p. 388]. It follows that the lattice 3_{31} consists of *all the points whose coordinates are* 8 *integers, mutually congruent* mod 2, *with sum* 0 [11, pp. 390–391, 403].

Among the points of this lattice, the 126 nearest to the origin are the 126 vertices of the vertex figure 2_{31}, namely *the* 56 *permutations of* (2, 0, 0, 0, 0, 0, 0, −2) *and the* 70 *permutations of* (1, 1, 1, 1, −1, −1, −1, −1).

Since the vertex figure of 2_{31} is $1_{31}=\mathrm{h}\gamma_6$, whose group is D_6, this number 126 arises as the index of $D_6=[3^{3,1,1}]$ in $E_7=[3^{3,2,1}]$. Similarly, 2_{31} has $|E_7|/|E_6|=56$ facets 2_{21} and $|E_7|/|A_6|=576$ facets $2_{30}=\alpha_6$. Since $\mathrm{h}\gamma_6$ has 32 vertices while its facets include 12 1_{21}'s, each vertex of 2_{31} belongs to 32 edges and 12 2_{21}'s. Similarly, since the 'second vertex figure' of 2_{31} is $0_{31}=\mathrm{t}_1\alpha_5$, whose facets consist of 6 $\mathrm{t}_1\alpha_4$'s and 6 α_4's, each edge of 2_{31} belongs to 6 2_{21}'s and 6 α_6's.

J.W.P. Hirschfeld [38, p. 120] has discovered, in projective 3-space, a configuration of 56+576 lines, 126 cubic curves and 2016 cubic surfaces, whose automorphism group is E_7. In fact, there is a perfect correspondence between his configuration and the polytope 2_{31}, as follows:

The polytope 2_{31}	Hirschfeld's configuration
126 vertices such as (2, −2, 0, 0, 0, 0, 0, 0) and (1, 1, 1, 1, −1, −1, −1, −1), each belonging to 32 edges and 12 facets 2_{21}.	126 twisted cubics such as t_1^2 and t_{1234}, each lying on 32 cubic surfaces and each having 12 Latin secants.
2160 edges, each joining 2 vertices and belonging to 6 2_{21}'s and 6 α_6's.	2160 cubic surfaces, each containing 2 cubic curves and a double six: 6 Latin lines and 6 Greek lines.
56 facets 2_{21} (lying in hyperplanes such as $u_1+u_2=2$ and $u_1+u_2=-2$), each having 27 vertices while each is adjacent to 27 other 2_{21}'s and 72 simplexes α_6.	56 Latin lines, such as A_{12} and B_{12}, each being a secant to 27 cubic curves while each intersects 27 other Latin lines and 72 Greek lines.
576 facets α_6 (lying in hyperplanes such as $u_1=2$, $u_1=-2$, $u_1-u_2-u_3-u_4=4$ and $u_2+u_3+u_4-u_1=4$), each having 21 edges while each is adjacent to 7 facets 2_{21}.	576 Greek lines (such as Γ_1, Δ_1, Γ_{234}^1 and Δ_{234}^1), each lying on 21 cubic surfaces while each intersects 7 Latin lines.

The finite subgroup E_7 of $\tilde{E}_7$ is generated by the 7 reflections

$$(1\ 2),\quad (2\ 3),\quad (3\ 4),\quad (4\ 5),\quad (5\ 6),\quad (6\ 7),$$
$$[1234\cdot 5678].$$

Their mirrors form a spherical 6-simplex in which the vertex opposite to the first mirror is given by

$$u_1 \mp u_2 = u_3 = \dots u_7, \qquad u_1+u_2+u_3+u_4=u_5+u_6+u_7+u_8=0.$$

Thus a typical vertex of the 7-dimensional polytope 3_{21} is $(-3, 1, 1, 1, 1, 1, 1, -3)$, and the whole set of 56 vertices consists of *the* $28+28$ *permutations of*

$$(-3, -3, 1, 1, 1, 1, 1, 1) \quad \textit{and} \quad (3, 3, -1, -1, -1, -1, -1, -1)$$

[10; 11, p. 387], which are conveniently denoted by

$$c_{12} \quad \text{and} \quad C_{12}.$$

In fact, C_{hi} is the $P_{h,i}$ of J.J. Seidel [2, pp. 303–304].

Altogether, the $8 \times 9!$ elements of the group E_7 include 63 reflections (interchanging the 63 pairs of opposite vertices of 2_{31}, and the 63 pairs of opposite β_6's of 3_{21} [14, p. 612]): the $\binom{8}{2}=28$ transpositions such as (1 2), and the $\frac{1}{2}\binom{8}{4}=35$ bifid substitutions. We easily verify that $[efgh \cdot ijkl]$ reflects c_{ef} into C_{gh}, c_{ij} into C_{kl}, while leaving c_{ei} and C_{ei} invariant. It is interesting to compare this with the $P_{efgh}=P_{ijkl}$ of [45, p. 364].

We see from § 2.5 (on page 573) that the $\binom{8}{2}$ vertices of $t_1\alpha_7$ are given by the permutations of two coordinates 6 followed by six coordinates -2. These can be halved to yield $(3, 3, -1, -1, -1, -1, -1, -1)$ or C_{12}; thus the 56 vertices of 3_{21} belong to two concentric $t_1\alpha_7$'s. Another way to exploit the central symmetry of 3_{21} is to observe that, since the vertex figure 2_{21} has 27 vertices, the 56 can be distributed as $1+27+27+1$: two opposite vertices, say

$$C_{78}=(-1, -1, -1, -1, -1, -1; 3, 3) \quad \text{and} \quad c_{78}=(1, 1, 1, 1, 1, 1; -3, -3),$$

and the vertices of two 2_{21}'s lying in parallel 6-flats

$$u_1+u_2+\dots u_6=\mp 2, \qquad u_7+u_8=\pm 2$$

between them. The 2_{21} in the former 6-flat (with the upper signs) has the $12+15$ vertices

$$(3, -1, -1, -1, -1, -1; 3, -1) \quad \text{and} \quad (-3, -3, 1, 1, 1, 1; 1, 1)$$

(with the first six coordinates permuted, and likewise the last two), i.e.,

$$C_{i7}, \quad C_{i8} \quad (1 \leqq i \leqq 6) \quad \text{and} \quad c_{ij} \quad (1 \leqq i < j \leqq 6). \tag{3.73}$$

These perfectly agree with Schläfli's symbols for the 27 lines on the cubic surface if we identify C_{i7} with his a_i and C_{i8} with his b_i [11, p. 388 (9.42)].

Like the 4 diameters of the cube in ordinary space, the 28 diameters of 3_{21} (joining pairs of opposite vertices c_{ij}, C_{ij}) are a set of *equiangular* lines [41, pp. 344–346]: all the $\binom{28}{2}$ pairs of diameters are congruent. Moreover, the two supplementary angles formed by any pair are arc sec(± 3), the same as in the case of the cube. This observation suggests that certain tetrads of diameters of 3_{21} may belong to cubes. In fact, 105 3-flats such as

$$u_1 = u_2, \quad u_3 = u_4, \quad u_5 = u_6, \quad u_7 = u_8$$

contain cubes, and so also do 210 such as

$$u_1 + u_3 = u_2 + u_4, \quad u_5 = u_6 = u_7 = u_8:$$

a total of 315 cubes whose vertices and edges all belong to 3_{21}.

When the 28 diameters of 3_{21} are represented by the 28 bitangents of a non-singular quartic curve in a projective plane [10; 45, pp. 351–362], the 63 *Steiner sets* represent the 63 pairs of opposite facets $3_{11} = \beta_6$, the 288 *Aronhold sets* represent the 288 pairs of opposite facets $3_{20} = \alpha_6$, and the 315 *conics* (each containing the 8 points of contact of 4 bitangents) represent the 315 cubes. The vertex figure of such a cube is one of the 45 "equatorial triangles" of 2_{21} (the vertex figure of 3_{21}) which are represented by the 45 tritangent planes of the cubic surface. As each of the 27 lines belongs to 5 tritangent planes, and $12 \times 315 = 5 \times 756$, each of the 756 edges of 3_{21} belongs to 5 of the 315 cubes.

By applying to such a cube any element of period 7 in E_7, we obtain a set of 7 cubes whose 7×8 vertices are precisely all the 56 vertices of 3_{21}. This arrangement corresponds to a set of 7 conics whose 7×8 intersections with the quartic curve are the points of contact of the 28 bitangents [24]. Patrick Du Val [13, p. 186; 41, p. 344] proved that such a set of 7 cubes can be derived from

$$(\pm 1, \pm 1, 0, \pm 1, 0, 0, 0) \tag{3.74}$$

by *cyclic* permutation of the Cartesian coordinates $x_0, x_1, \ldots, x_6$. We recall [21, p. 244] that the $2n$ vertices of the regular cross polytope β_n can be orthogonally projected into the vertices of a regular polygon $\{2n\}$ is a suitable plane, which we may identify with the field of complex numbers. Thus, when $n = 7$, the point $(x_0, x_1, \ldots, x_6)$ appears as the complex number

$$\sum_{\nu=0}^{6} x_\nu \varepsilon^\nu, \quad \text{where} \quad \varepsilon = e^{2\pi i/7},$$

the 14 vertices of β_7 are given by $\pm \varepsilon^\mu$, and the 56 vertices of 3_{21} are given by

$$\pm \varepsilon^\mu \pm \varepsilon^{\mu+1} \pm \varepsilon^{\mu+3} \quad (\mu = 0, 1, \ldots, 6)$$

with one cube for each value of μ. Fig. 3.7a shows these 56 points and the edges of one of the seven cubes. The remaining six cubes can be found by rotating the figure through multiples of the angle

$$\theta = 2\pi/7.$$

These rotations permute the residues μ (mod 7) which appear in the abbreviated symbols 013 for $1 + \varepsilon + \varepsilon^3$, $2\overline{3}\overline{5}$ for $\varepsilon^2 - \varepsilon^3 - \varepsilon^5$, and so on.

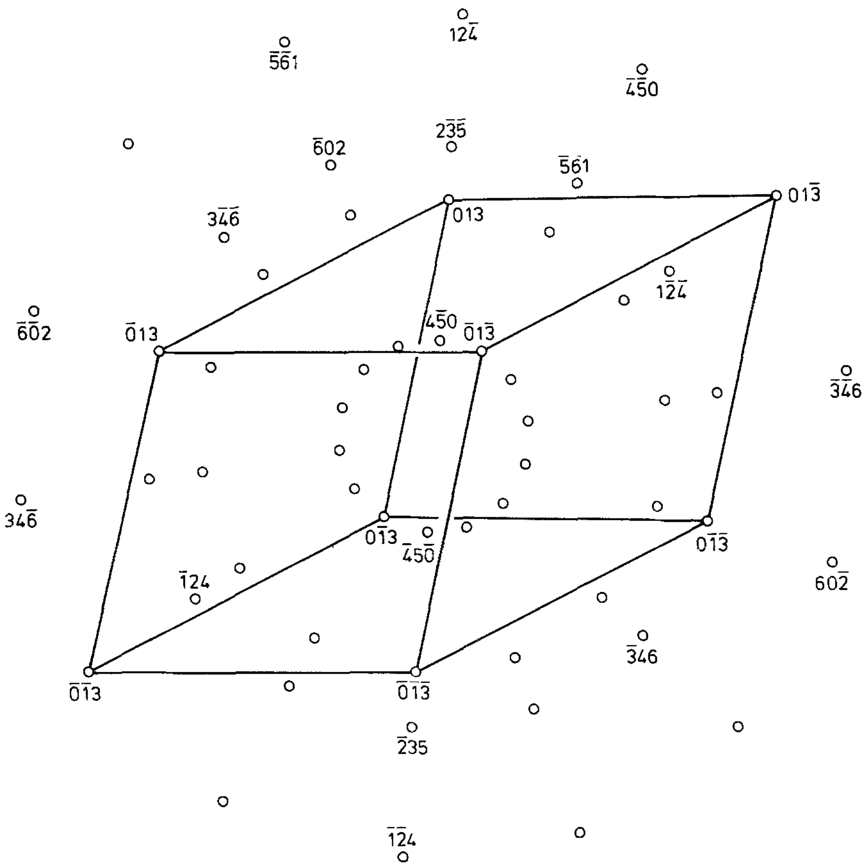


Fig. 3.7a. The 56 vertices of 3_{21} and one of the 315 inscribed cubes

Remarkably, the 56 points in this figure lie in sets of 8 on 7 lines through the centre 0. To see this, we observe that $2\bar{3}\bar{5}$, $\bar{4}5\bar{0}$, $\bar{1}\bar{2}4$ are collinear with 0 and 013, since

$$\begin{aligned}
2\cos\theta\cdot(1+\varepsilon+\varepsilon^3) &= (\varepsilon\ +\varepsilon^{-1})(1+\varepsilon+\varepsilon^3) = \ \varepsilon^2-\varepsilon^3-\varepsilon^5 = 2\bar{3}\bar{5},\\
2\cos 2\theta\cdot(1+\varepsilon+\varepsilon^3) &= (\varepsilon^2+\varepsilon^{-2})(1+\varepsilon+\varepsilon^3) = -\varepsilon^4+\varepsilon^5-1 = \bar{4}5\bar{0},\\
2\cos 3\theta\cdot(1+\varepsilon+\varepsilon^3) &= (\varepsilon^3+\varepsilon^{-3})(1+\varepsilon+\varepsilon^3) = -\varepsilon-\varepsilon^2+\varepsilon^4 = \bar{1}\bar{2}4.
\end{aligned}$$

Moreover, each of these sets of 8 collinear points comes from the vertices of a cube, as we see from Fig. 3.7b, where the vertices have been artificially separated for clarity.

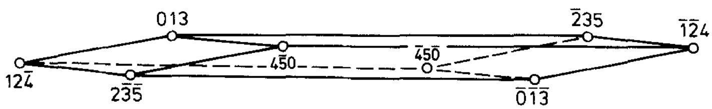


Fig. 3.7b. Another cube, foreshortened

Thus the 56 points in Fig. 3.7a form four concentric {14}'s:

$$12\bar{4}\quad \bar{5}61\quad 23\bar{5}\quad \bar{6}02\quad 34\bar{6}\quad \bar{0}13\quad 45\bar{0}\quad \bar{1}24\quad 56\bar{1}\quad \bar{2}\bar{3}5\quad 60\bar{2}\quad \bar{3}46\quad 01\bar{3}\quad \bar{4}50,$$
$$2\bar{3}\bar{5}\quad \bar{6}02\quad 3\bar{4}\bar{6}\quad \bar{0}13\quad 4\bar{5}\bar{0}\quad \bar{1}24\quad 5\bar{6}\bar{1}\quad \bar{2}35\quad 6\bar{0}\bar{2}\quad \bar{3}46\quad 0\bar{1}\bar{3}\quad \bar{4}50\quad 1\bar{2}\bar{4}\quad \bar{5}61,$$
$$013\quad \bar{4}\bar{5}0\quad 124\quad \bar{5}\bar{6}1\quad 235\quad \bar{6}\bar{0}2\quad 346\quad \bar{0}\bar{1}3\quad 450\quad \bar{1}\bar{2}4\quad 561\quad \bar{2}\bar{3}5\quad 602\quad \bar{3}\bar{4}6,$$
$$4\bar{5}0\quad \bar{1}2\bar{4}\quad 5\bar{6}1\quad \bar{2}3\bar{5}\quad 6\bar{0}2\quad \bar{3}4\bar{6}\quad 0\bar{1}3\quad \bar{4}5\bar{0}\quad 1\bar{2}4\quad \bar{5}6\bar{1}\quad 2\bar{3}5\quad \bar{6}0\bar{2}\quad 3\bar{4}6\quad \bar{0}1\bar{3}.$$

After drawing the first regular 14-gon, we can locate the vertices of the remaining three as the points of intersection of certain pairs of diagonals. Each of these diagonals contains a set of four collinear points, arising from a foreshortened square. To be precise, $2\bar{3}\bar{5}$ is at the intersection of diagonals $\bar{5}61$ $\bar{3}46$ and $\bar{6}\bar{0}2$ $\bar{4}\bar{5}0$; 013, of $\bar{5}\bar{6}1$ $60\bar{2}$ and $34\bar{6}$ $\bar{4}\bar{5}0$ (or of $\bar{6}02$ $1\bar{2}\bar{4}$ and $3\bar{4}\bar{6}$ $\bar{5}61$); $4\bar{5}0$, of $\bar{6}02$ $\bar{3}46$ and $\bar{1}24$ $\bar{5}61$. With this information it would be easy to draw the whole 14-gonal projection of 3_{21}.

To find coordinates for the 126 vertices of the related polytope 2_{31}, we can identify the centres of the 126 facets $3_{11}=\beta_6$ of 3_{21} with the midpoints of the 'first' diagonals, of length 4. Since the vectors obtained by adding (1, 1, 0, 1, 0, 0, 0) to

$$\begin{array}{lll}(1,-1,0,-1,0,0,0), & (-1,1,0,-1,0,0,0), & (-1,-1,0,1,0,0,0),\\ (0,-1,1,0,1,0,0), & (0,0,1,-1,0,1,0), & (0,0,0,-1,1,0,1),\\ (-1,0,0,0,1,1,0), & (0,-1,0,0,0,1,1), & (-1,0,1,0,0,0,1)\end{array}$$

in turn, are

$$\begin{array}{lll}(2,0,0,0,0,0,0), & (0,2,0,0,0,0,0), & (0,0,0,2,0,0,0),\\ (1,0,1,1,1,0,0), & (1,1,1,0,0,1,0), & (1,1,0,0,1,0,1),\\ (0,1,0,1,1,1,0), & (1,0,0,1,0,1,1), & (0,1,1,1,0,0,1),\end{array}$$

the $14+112$ vertices of 2_{31} can be derived from

$$(\pm 2,0,0,0,0,0,0) \quad \text{and} \quad (\pm 1,\pm 1,\pm 1,0,0,\pm 1,0) \tag{3.75}$$

by cyclic permutations of the 7 coordinates.

It follows that the vertices of the lattice 3_{31} are *all the points whose seven coordinates are even, along with the cyclic permutations of*

$$(1,1,1,0,0,1,0) \quad (\text{mod } 2) \tag{3.76}$$

[47, pp. 148–149 ('Example 8')]. The vertices (3.75) of 2_{31} are just the 126 lattice points at distance 2 from the origin. Similarly, the 576 lattice points at distance $\sqrt{7}$ consist of the 128 points

$$(\pm 1,\pm 1,\pm 1,\pm 1,\pm 1,\pm 1,\pm 1)$$

and the 448 cyclic permutations of

$$\begin{array}{ll}(\pm 1,\pm 1,\pm 2,\pm 1,0,0,0), & (\pm 1,\pm 1,0,\pm 1,\pm 2,0,0),\\ (\pm 1,\pm 1,0,\pm 1,0,\pm 2,0), & (\pm 1,\pm 1,0,\pm 1,0,0,\pm 2).\end{array}$$

These 576 can be identified with the centres of the facets $3_{20}=\alpha_6$ of 3_{21}, such as the simplex

$$(1,1,0,1,0,0,0) \quad (0,1,1,0,\pm 1,0,0) \quad (0,0,1,1,0,\pm 1,0) \quad (1,0,1,0,0,0,\pm 1)$$

with centre $\frac{3}{7}(1, 1, 2, 1, 0, 0, 0)$. In other words, *the* 576 *points named above are the vertices of the polytope* 1_{32}.

Like the 'extended' icosahedral group $[3, 5] \cong \mathfrak{A}_5 \times \mathfrak{S}_2$ (see § 2.1 on page 561), E_7 includes the central inversion whereas its 'even' subgroup E_7^+ does not; therefore

$$E_7 \cong E_7^+ \times \mathfrak{S}_2,$$

where E_7^+ is the simple group of order $4 \times 9! = 1451520$, namely

$$\mathrm{Sp}_6(2) \cong \mathrm{S}_6(2) \cong \mathrm{GO}_7(2) \cong \mathrm{SO}_7(2) \cong \mathrm{O}_7(2)$$

[9, p. 46].

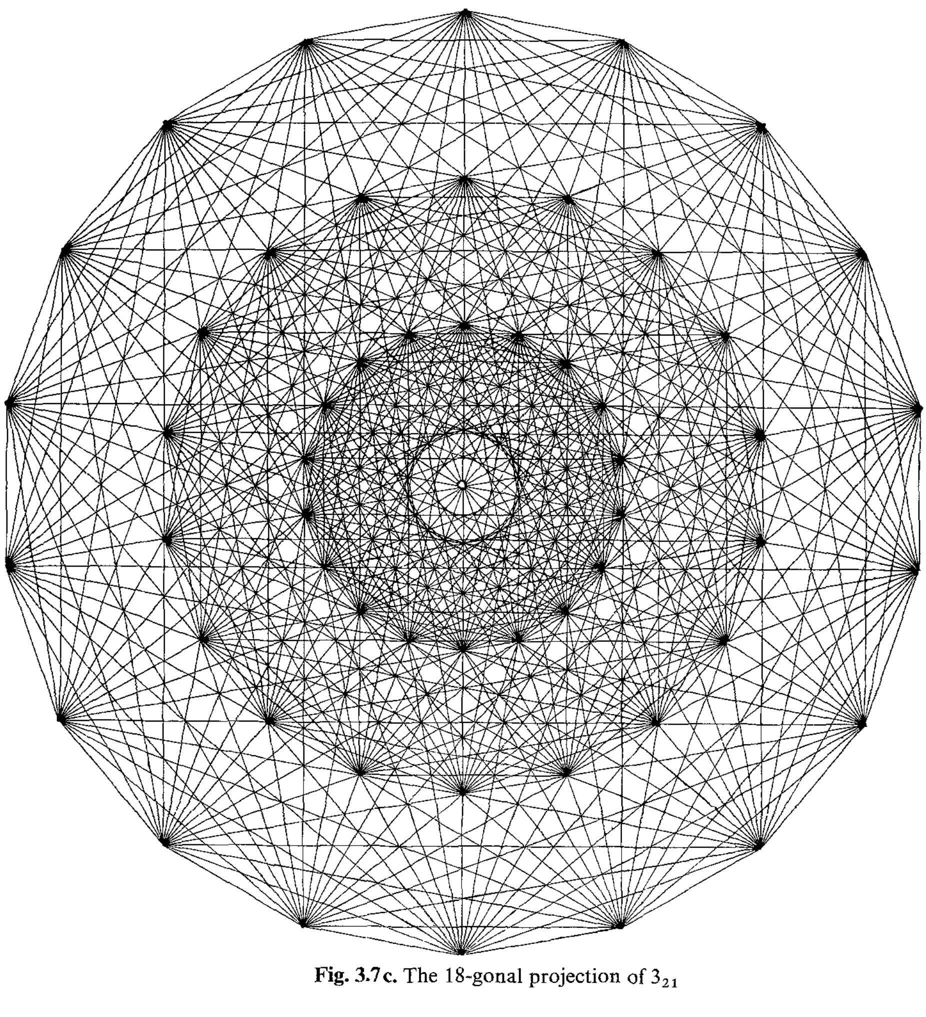
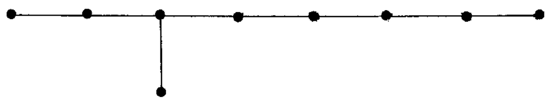


Fig. 3.7c. The 18-gonal projection of 3_{21}

Fig. 3.7c is Peter McMullen's drawing, in which the 56 vertices of 3_{21} appear as three concentric $\{18\}$s with one pair of opposite vertices coinciding in the centre. This is even more symmetrical than the 14-gonal projection that arose from the compound of seven cubes.

3.8. The Groups E_8, $\tilde{E}_8 = E_9$ and Lattice 5_{21}

The infinite group $\tilde{E}_8 = E_9 = [3^{5,2,1}]$ arises from the kaleidoscope



whose 9 mirrors may be constructed in Euclidean 9-space as sections of the 9 hyperplanes

$$u_1 = u_2,\quad u_2 = u_3,\quad u_3 = u_4,\quad u_4 = u_5,\quad u_5 = u_6,\quad u_6 = u_7,\quad u_7 = u_8,\quad u_8 = u_9 + 3,$$
$$2(u_1 + u_2 + u_3) = u_4 + u_5 + u_6 + u_7 + u_8 + u_9 \tag{3.81}$$

by the hyperplane $\sum u_\nu = 0$. Thus, in the notation of page 17, r_1 is (3.81), r_2 is $u_1 = u_2$, r_3 is $u_2 = u_3$, ..., r_8 is $u_7 = u_8$, and r_9 is $u_8 = u_9 + 3$ [28, p. 29]. Moreover, the hyperplane s, bisecting the right angle between r_1 and r_3, is $3(u_2 - u_3) + 2(u_1 + u_2 + u_3) - (u_4 + u_5 + \ldots u_9) = 0$ or

$$2u_1 + 5u_2 - (u_3 + u_4 + \ldots + u_9) = 0;$$

and t, perpendicular to r_3 through the intersection of r_2 and s, is $6(u_1 - u_2) + 2u_1 + 5u_2 - (u_3 + \ldots + u_9) = 0$ or

$$8u_1 - (u_2 + u_3 + \ldots + u_9) = 0.$$

These hyperplanes s and t are perpendicular to r_ν for all $\nu > 3$, while t is perpendicular also to r_3; thus they decompose the 8-simplex $r_1 r_2 \ldots r_9$ into three orthoschemes

$$r_2 s r_1 r_4 \ldots r_9,\qquad t s r_3 r_4 \ldots r_9,\qquad t r_2 r_3 r_4 \ldots r_9,$$

the first two of which are congruent. We observe that s makes equal angles $\pi/4$ with r_3 and r_9, and equal angles σ with t and r_2, where

$$\cos\sigma = (16 - 5 + 7)/6 \times 6\sqrt{2} = (-2 + 5)/6\sqrt{2} = 1/2\sqrt{2}.$$

Also, in the last orthoscheme, the angle ρ between t and r_2 is given by

$$\cos\rho = (8 + 1)/6\sqrt{2} \times \sqrt{2} = 3/4,$$

in agreement with (3.51). Thus the decomposition which led us to (3.55) remains valid when $n = 9$, and all the cases with $n < 9$ could have been deduced from this Euclidean case.

In the 8-simplex $r_1 r_2 \dots r_9$, the vertex opposite to r_9 is given by solving all the ten equations except $u_8 = u_9 + 3$; thus it is the origin, whose image by reflection in r_9 is

$$(0, 0, 0, 0, 0, 0, 0, 3, -3).$$

Since the mirror r_1 or (3.81) reflects $(0, 0, 3, 0, 0, 0, 0, 0, -3)$ into

$$(-2, -2, 1, 1, 1, 1, 1, 1, -2),$$

we conclude that the lattice 5_{21} consists of *all the points whose coordinates are* 9 *integers, mutually congruent* mod 3, *with sum zero* [11, p. 402].

Similarly, omitting $u_1 = u_2$ and doubling, we obtain the typical vertex

$$(-2, 1, 1, 1, 1, 1, 1, 1, -5)$$

of the related honeycomb 2_{51}. Its vertices are a subset of the lattice, namely *those points for which* 4 *or* 8 *of the* 9 *coordinates are odd* (as 1 and -5 are in the above instance).

As we know, the *complex number* $a + bi$ is derived from two real numbers, a and b, by means of the multiplication rule

$$(a+bi)(c+di) = ac - db + (da + bc)\,i.$$

The complex conjugate of $x = a + bi$ is $\bar{x} = a - bi$.

The *quarternion* $a + bj$ is derived from two complex numbers, a and b, by the rule

$$(a+bj)(c+dj) = ac - \bar{d}b + (da + b\bar{c})\,j.$$

The quarternation conjugate of $x = a + bj$ is $\tilde{x} = \bar{a} - bj$.

The *octave* (or Cayley number) $a + be$ is derived from two quarternions, a and b, by the rule

$$(a+be)(c+de) = ac - \tilde{d}b + (da + b\tilde{c})\,e \tag{3.82}$$

[27, p. 158; 19, pp. 27–29]. Writing

$$a = a_0 + a_1 i + a_2 j + a_3 k \quad \text{and} \quad b = b_0 + b_1 i + b_2 j + b_3 k,$$

where a_ν and b_ν are real numbers and $k = ij$, we have

$$a + be = a_0 + a_1 i + a_2 j + a_3 k + b_0 e + b_1 ie + b_2 je + b_3 ke,$$

where e, ie, je, ke, like i, j, k, are square roots of -1.

Such combinations of 8 real numbers were discovered in 1843 by J.T. Graves, who proposed (in a letter of 1844 to Hamilton) the name *octaves*. However, they are often called *Cayley numbers* because, in 1847, Cayley noticed that for some triads abc, the associative property $ab \cdot c = a \cdot bc$ must be replaced by the anti-associative property $ab \cdot c = -a \cdot bc$ [4, p. 106; 8, p. 301]. Dickson's seven symbols

$$i,\ j,\ k,\ e,\ ie,\ je,\ ke$$

can be applied to the 7 points of Fano's finite projective plane $PG(2, 2)$ as in Fig. 3.8a, so that the 7 lines (one drawn as a circle) determine *associative* triads abc, such as ijk, whose pairs anti-commute:

$$bc=a=-cb, \quad ca=b=-ac, \quad ab=c=-ba.$$

The associative law is easily seen to hold also for any triad with a *repeated* element: $a^2b=a\cdot ab$, $ab\cdot a=a\cdot ba$, $ab\cdot b=ab^2$. The 7 'complements' of the lines in the finite geometry are quadrangles $abcd$, such as $e\ ie\ je\ ke$, which (when arranged in a suitable order) satisfy

$$bc\cdot d=a=-b\cdot cd, \quad c\cdot da=b=-cd\cdot a,$$
$$da\cdot b=c=-d\cdot ab, \quad a\cdot bc=d=-ab\cdot c.$$

In other words, the 28 triangles that occur in the 7 quadrangles determine the 28 *anti-associative* triads [19, p. 23]. For instance,

$$(ie\cdot je)\,ke=(ji)\,ke=(-k)\,ke=e,$$
$$ie(je\cdot ke)=ie(kj)=ie(-i)=iie=-e.$$

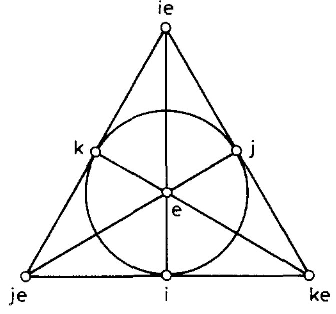


Fig. 3.8a. A finite plane with seven points and seven lines

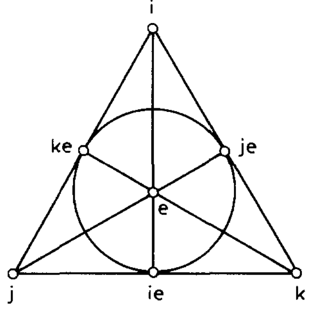


Fig. 3.8b. The same seven points differently named

Fig. 3.8b shows a different way to apply the same 7 symbols to the points of $PG(2, 2)$. Now the 7 points, 7 lines (along with ± 1) and 7 quadrangles represent, in an equally perspicuous manner, the $16+16+16$ octaves

$$\pm 1, \pm i, \pm j, \pm k, \pm e, \pm ie, \pm je, \pm ke,$$
$$(\pm 1 \pm ie \pm je \pm ke)/2, \quad (\pm e \pm i \pm j \pm k)/2$$

and 192 others derived from the last two expressions by cyclically permuting the 7 symbols in the peculiar order

$$e,\ i,\ j,\ ie,\ ke,\ k,\ je.$$

These 16+224 octaves have the remarkable property of being closed under multiplication. For instance,

$$\begin{aligned}(1+ie+je+ke)(1+ke+e+k) &= \{1+(i+j+k)\,e\}\,\{(1+k)+(1+k)\,e\}\\ &=(1+k)-(1-k)(i+j+k)+\{(1+k)+(i+j+k)(1-k)\}\,e\\ &=2(-i+e+je+ke).\end{aligned}$$

Accordingly, they are said to be the 240 *units* in a maximal domain of *integral* octaves. The whole integral domain (based on these units) consists of the octaves

$$(x_0+x_1\,e+x_2\,i+x_3\,j+x_4\,ie+x_5\,ke+x_6\,k+x_7\,je)/2,$$

where *the eight x's are ordinary integers, all even or all odd, or four of each, with a restriction in the last case: their residues* mod 2 *may only be*

$$(1;0,0,0,1,1,0,1) \quad \textit{or} \quad (0;1,1,1,0,0,1,0)$$

or the same with the last seven cyclically permuted.

Interpreting $(x_0, x_1, \ldots, x_7)$ as a point in Euclidean 8-space, we recognize this domain of integral octaves as one of Patrick Du Val's coordinatizations for the lattice 5_{21} [13, pp. 185–186]. Its section by the hyperplane $x_0=0$ is the lattice 3_{31}; see (3.76).

It seems somewhat paradoxical [5, p. 128] that the cyclic permutation

$$(e\;\; i\;\; j\;\; ie\;\; ke\;\; k\;\; je),$$

which preserves the integral domain (and the finite projective plane labelled as in Fig. 3.8b), is not an automorphism of the whole ring of octaves: it transforms the associative triad ijk into the anti-associative triad $j\;ie\;je$. On the other hand, the permutation

$$(e\;\; ie\;\; je\;\; i\;\; k\;\; ke\;\; j),$$

which *is* an automorphism of the whole ring of octaves (and of the finite plane labelled as in Fig. 3.8a) transforms this particular integral domain into another one of R.H. Bruck's cyclic of seven such domains [19, p. 27].

The 240 units of the integral domain represent the 240 lattice points at distance 2 from the origin: the 16 permutations of

$$(\pm 2, 0, 0, 0, 0, 0, 0, 0)$$

and the 112+112 cyclic permutations of (the last 7 coordinates in)

$$(\pm 1; 0, 0, 0, \pm 1, \pm 1, 0, \pm 1), \qquad (0; \pm 1, \pm 1, \pm 1, 0, 0, \pm 1, 0). \tag{3.83}$$

These 240 points are the vertices of the 8-dimensional uniform polytope 4_{21}, which has also 6720 edges, 60480 triangular faces, 241920 tetrahedra, 483840 four-dimensional simplexes, the same number of five-dimensional simplexes 4_{00}, 138240+69120 six-dimensional simplexes 4_{10} and 4_{01}, and, for its facets, 17280 seven-dimensional simplexes 4_{20} along with 2160 cross polytopes 4_{11} [11,

p. 414]. Fig. 3.8c is Peter McMullen's skilful drawing of its most symmetrical planar projection, in which the 240 vertices are distributed into 8 concentric triacontagons $\{30\}$. This drawing could have been derived from the frontispiece of *Regular Complex Polytopes* by inserting 240 extra lines [22, p. 134].

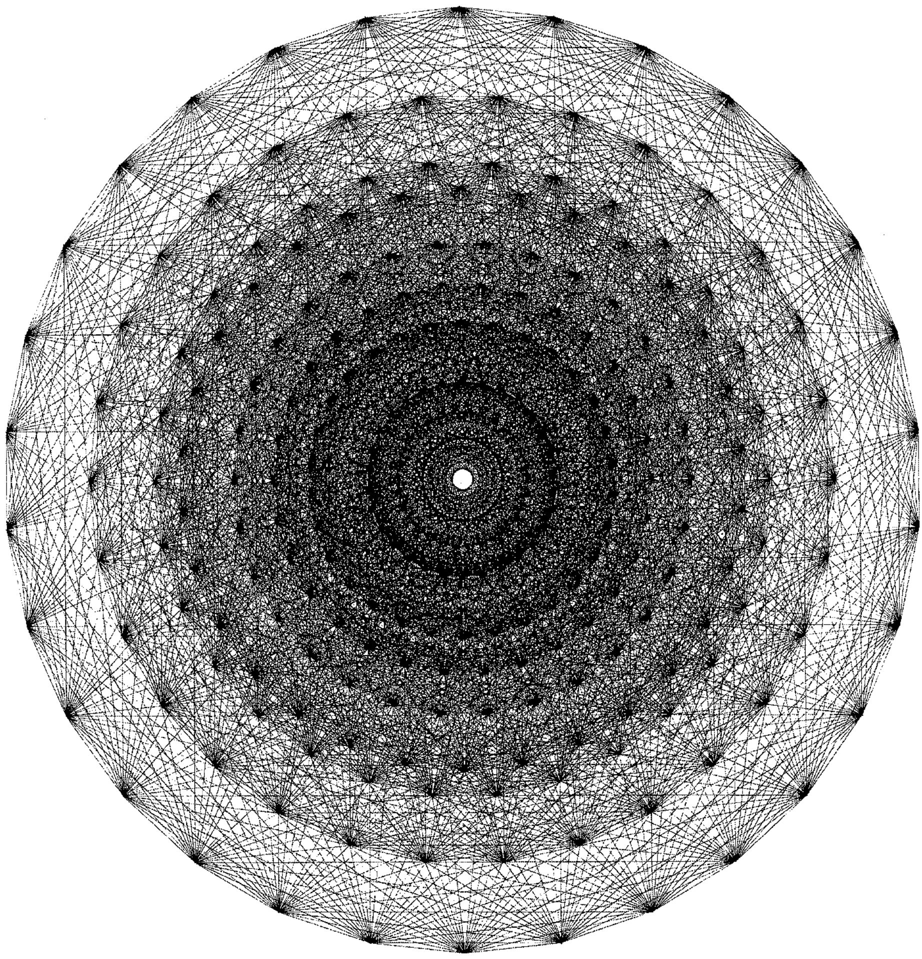


Fig. 3.8c. The eight-dimensional polytope 4_{21}

Returning to the lattice 5_{21}, we may consider the points at distance $2\sqrt{2}$ from the origin [19, p. 35] and thus obtain, for the $112+256+1792$ vertices of 2_{41},

$$(\pm 2, \pm 2, 0, 0, 0, 0, 0, 0), \qquad (\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$$

and (3.83) with each 0, in turn, replaced by ± 2. The supporting hyperplane $x_0 = 2$ contains one of the 240 facets 2_{31}, with the coordinates (3.75).

The same lattice 5_{21} includes 17520 points at distance 4 from the origin, namely: the $16+1120+2048$ permutations of

$$(\pm 4, 0, 0, 0, 0, 0, 0, 0), \quad (\pm 2, \pm 2, \pm 2, \pm 2, 0, 0, 0, 0),$$
$$(\pm 3, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1),$$

the 7168 cyclic permutations of (3.83) with three 0's replaced by ± 2, and the 7168 cyclic permutations of (3.83) with one 0 replaced by ± 2 and one ± 1 replaced by ± 3. These 17520 points include the 240 vertices of a dilated 4_{21} (with all the coordinates doubled). When these 240 have been removed [11, p. 397–398; 19, pp. 35–36], *the remaining* 17280 *are the vertices of* 1_{42}.

M.S. Longuet-Higgins [42, pp. 450–466] has discovered, in inversive 3-space, a configuration of 17280 points and 2160 spheres, with 8 spheres through each point and 64 points on each sphere; its automorphism group is $[3^{4,2,1}]$. In fact, the points and spheres correspond to the vertices and facets $1_{41}=\mathrm{h}\gamma_7$ of the polytope 1_{42}, in which each vertex belongs to 8 such facets while each $\mathrm{h}\gamma_7$ has 64 vertices. He exhibits this configuration as one member of a large family. For any two positive integers q and r, with $(q-1)(r-1)\leqq 3$, inversive $(q+1)$-space admits a configuration of points and q-spheres corresponding to the vertices and facets $1_{(q-1)r}$ of the $(q+r+2)$-dimensional polytope 1_{qr} $(=1_{rq})$ which can be defined simply as the polytope whose second vertex figure is the Cartesian product $\alpha_q \times \alpha_r$ of two regular simplexes. There are $q+r+2$ q-spheres through each point, arising from the $q+r+2$ facets $0_{(q-1)r}$ of the vertex figure $0_{qr}=\mathrm{t}_q\,\alpha_{q+r+1}$. The points on each q-sphere correspond to the vertices of the facet $1_{(q-1)r}$. The case when $q=1$ was already mentioned on page 7, where we considered the planar configuration of 2^{r+2} points and 2^{r+2} circles corresponding to the vertices and facets α_{r+2} of the $(r+3)$-dimensional hemi-cube $1_{1r}=\mathrm{h}\gamma_{r+3}$.

Du Val, whose eight-dimensional coordinates for the lattice 5_{21} are related to the octaves, discovered also ten-dimensional coordinates for the same lattice [13, pp. 186–187]. In fact, the vertices of a 5_{21} of edge $5\sqrt{2}$ are all the points in Euclidean 10-space whose coordinates satisfy the equations

$$x_1+x_2+x_3+x_4+x_5=x_6+x_7+x_8+x_9+x_{10}=0$$

and the congruences

$$x_1 \equiv x_2 \equiv x_3 \equiv x_4 \equiv x_5 \equiv 2x_6 \equiv 2x_7 \equiv 2x_8 \equiv 2x_9 \equiv 2x_{10} \pmod 5.$$

In this lattice, the points at distance $5\sqrt{2}$ from the origin are, of course, the 240 vertices of a 4_{21}. In the accompanying table, we allow *simultaneous* permutations of the first five and last five coordinates. On the left we display the x's, and on the right, new coordinates

$$u_\nu = \begin{cases} (x_\nu + \tau x_{\nu+5})/\sqrt{5} & (\nu=1,2,3,4,5), \\ (\tau x_{\nu-5} - x_\nu)/\sqrt{5} & (\nu=6,7,8,9,10), \end{cases}$$

where $\tau=\frac{1}{2}(\sqrt{5}+1)$.

Du Val's coordinates	New coordinates
$(5, 0, 0, 0, -5; 0, 0, 0, 0, 0)$	$(\sqrt{5}, 0, 0, 0, -\sqrt{5}; \tau\sqrt{5}, 0, 0, 0, -\tau\sqrt{5})$
$(4, -1, -1, -1, -1; 2, 2, 2, -3, -3)$	$(2\tau, 1, 1, -\tau^2, -\tau^2; 2, -\tau, -\tau, \tau^{-1}, \tau^{-1})$
$(4, -1, -1, -1, -1; -3, -3, 2, 2, 2)$	$(-\tau^{-2}, -\tau^2, 1, 1, 1; \tau^3, \tau^{-1}, -\tau, -\tau, -\tau)$
$(3, 3, -2, -2, -2; 4, -1, -1, -1, -1)$	$(\tau^3, \tau^{-1}, -\tau, -\tau, -\tau; \tau^{-2}, \tau^2, -1, -1, -1)$
$(3, 3, -2, -2, -2; -1, -1, -1, -1, 4)$	$(\tau^{-1}, \tau^{-1}, -\tau, -\tau, 2; \tau^2, \tau^2, -1, -1, -2\tau)$
$(2, 2, 2, -3, -3; 1, 1, 1, 1, -4)$	$(\tau, \tau, \tau, -\tau^{-1}, -\tau^3; 1, 1, 1, -\tau^2, -\tau^{-2})$
$(2, 2, 2, -3, -3; -4, 1, 1, 1, 1)$	$(-2, \tau, \tau, -\tau^{-1}, -\tau^{-1}; 2\tau, 1, 1, -\tau^2, -\tau^2)$
$(1, 1, 1, 1, -4; 3, 3, -2, -2, -2)$	$(\tau^2, \tau^2, -1, -1, -2\tau; -\tau^{-1}, -\tau^{-1}, \tau, \tau, -2)$
$(1, 1, 1, 1, -4; -2, -2, -2, 3, 3)$	$(-1, -1, -1, \tau^2, \tau^{-2}; \tau, \tau, \tau, -\tau^{-1}, -\tau^3)$
$(0, 0, 0, 0, 0; 5, 0, 0, 0, -5)$	$(\tau\sqrt{5}, 0, 0, 0, -\tau\sqrt{5}; -\sqrt{5}, 0, 0, 0, \sqrt{5})$

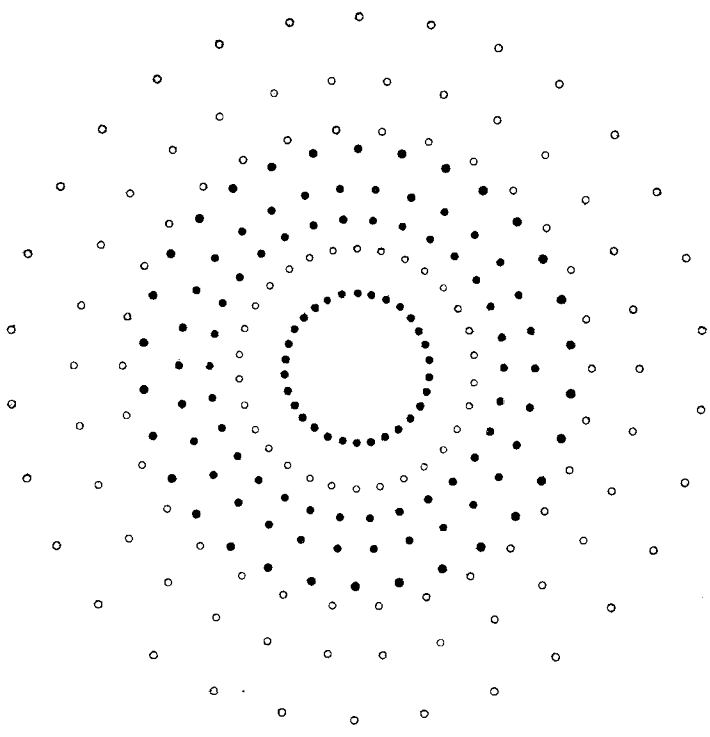


Fig. 3.8d. The $120+120$ vertices of the polytope 4_{21}

By picking out *alternate* rows of the right-hand column of the table, we distinguish two sets of 120 vertices of 4_{21} : one set satisfying

$$u_1^2+\ldots+u_5^2=10, \qquad u_6^2+\ldots+u_{10}^2=10\tau^2$$

and the other satisfying

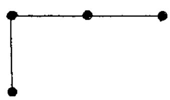
$$u_1^2+\ldots+u_5^2=10\tau^2, \qquad u_6^2+\ldots+u_{10}^2=10.$$

Let us call these 'odd' and 'even' vertices, respectively. In Fig. 3.8d they appear as black and white dots. (Compare Fig. 13.6A of [21, p. 249].) When we project onto the 5-space $u_6 = \dots = u_{10} = 0$ by ignoring the last five coordinates, we obtain the 120+120 vertices of two homothetic 600-cells $\{3, 3, 5\}$: one having the coordinates displayed in § 2.5 (on page 578) while the other has these same coordinates multiplied by τ. The dilatation that relates them is plainly visible in the figure.

Barry Monson [46] approaches this derivation (of two $\{3, 3, 5\}$'s from 4_{21}) by a different route. He observes that, in E_8, the four half-turns

$$\mathrm{R}_1\,\mathrm{R}_5, \quad \mathrm{R}_4\,\mathrm{R}_6, \quad \mathrm{R}_3\,\mathrm{R}_7, \quad \mathrm{R}_2\,\mathrm{R}_8$$

generate a subgroup isomorphic to [5, 3, 3]; for instance, $\mathrm{R}_1\,\mathrm{R}_5 \cdot \mathrm{R}_4\,\mathrm{R}_6$ (being a product of the four generators of the subgroup



which is $[3, 3, 3] \cong \mathfrak{S}_5$) has period 5. His conclusion is that 720 of the 6720 edges of 4_{21}, and 600 of the 241920 tetrahedra, project into the 720 edges and 600 facets of the larger ('even') 600-cell, while another set of 720 edges and 600 tetrahedra project into the edges and facets of the starry *grand 600-cell* $\{3, 3, \frac{5}{2}\}$ [22, p. 46] which has the same 120 vertices as the smaller ('odd') $\{3, 3, 5\}$. This remark agrees with the visible fact that the 'hole' in the middle of Fig. 3.8c resembles the hole in the middle of Figure 4.7C of *Regular Complex Polytopes* [22, p. 44] but is quite different from the hole in the middle of Figure 4.7A [22, p. 42]. The 600 tetrahedra in 4_{21} which project into the facets of the larger $\{3, 3, 5\}$, may be regarded as the facets of an 8-dimensional *skew* 600-cell whose 120 vertices are all the 'even' vertices of 4_{21}. This skew 600-cell inscribed in 4_{21} is somewhat analogous to the 6-dimensional 'skew icosahedron' inscribed in $\mathrm{h}\gamma_6$ (or 1_{31}) [19, p. 144, Fig. 25]. The subject of regular skew polytopes is undergoing further investigation by Peter McMullen and Egon Schulte.

3.9. The Exponents

This final section fulfils the promise, made at the end of § 3.3, to describe yet another method for computing the order of a finite reflection group. It makes use of the characteristic equation for the product of the n generating reflections, which can be multiplied in any order, since any two such products are conjugate [6, p. 117]. This so-called 'Coxeter transformation', being an isometry, is the product of rotations in $[\frac{1}{2}n]$ completely orthogonal planes along with an extra reflection if n is odd [21, pp. 221, 226, 234]. If $\mathrm{R}_1\,\mathrm{R}_2 \dots \mathrm{R}_n$ has period h, the rotations in the various planes are through angles which are multiples of $2\pi/h$, say

$$\xi_\nu = 2m_\nu\,\pi/h \qquad (\nu = 1, 2, \dots, [\tfrac{1}{2}n]).$$

Since the characteristic roots are $e^{\pm \xi_\nu i}$, along with $e^{\pi i} = -1$ if n is odd, the characteristic equation is

$$\prod_{\nu=1}^{n} (\lambda - e^{2 m_\nu \pi i/h}) = 0,$$

where

$$1 = m_1 \leqq m_2 \leqq \dots \leqq m_n = h-1 \quad \text{and} \quad m_{n+1-\nu} = h - m_\nu$$

so that, if n is odd, $m_{\frac{1}{2}(n+1)} = \frac{1}{2}h$. In other words, the n characteristic roots are

$$(e^{2\pi i/h})^{m_\nu} \qquad (\nu = 1, 2, \dots, n).$$

The *exponents* m_ν [6, pp. 118–210; 26, p. 212] have interesting properties, as we shall see later.

Instead of the characteristic equation of degree n in λ, we shall use a more convenient equation of degree $[\frac{1}{2}n]$ in

$$x = \lambda + \lambda^{-1} = 2 \cos \xi_\nu .$$

Most of the particular cases are neatly expressible in terms of the Chebyshev polynomials

$$T_n(X) = \frac{n}{2} \sum_{\nu=0}^{[n/2]} \frac{(-1)^\nu}{n-\nu} \binom{n-\nu}{\nu} (2X)^{n-2\nu},$$

$$U_n(X) = \sum_{\nu=0}^{[n/2]} (-1)^\nu \binom{n-\nu}{\nu} (2X)^{n-2\nu},$$

which are relevant because

$$T_n(\cos\theta) = \cos n\theta, \qquad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

Earlier treatments [21, pp. 220–234; 17, pp. 768–771; 32] used $X = \cos\theta$, where $\theta = \frac{1}{2}\xi_\nu$, instead of $x = 2\cos 2\theta$. To make the adaptation, we can substitute $\sqrt{x+2}$ for $2X$.

Since

$$U_{k-1}(\tfrac{1}{2}x) = \frac{\sin 2k\theta}{\sin 2\theta} = \frac{U_{2k-1}(X)}{2X}$$

and

$$\begin{aligned} U_k(\tfrac{1}{2}x) + U_{k-1}(\tfrac{1}{2}x) &= \{\sin(2k+2)\theta + \sin 2k\theta\}/\sin 2\theta \\ &= \sin(2k+1)\theta/\sin\theta = U_{2k}(X), \end{aligned}$$

the equation $U_n(X) = 0$ for A_n or $[3^{n-1}]$ becomes

$$U_{\frac{1}{2}(n-1)}(\tfrac{1}{2}x) = 0 \qquad \text{when } n \text{ is odd},$$

$$U_{\frac{1}{2}n}(\tfrac{1}{2}x) + U_{\frac{1}{2}n-1}(\tfrac{1}{2}x) = 0 \qquad \text{when } n \text{ is even}.$$

Since $T_k(\frac{1}{2}x) = T_{2k}(X)$ and

$$U_k(\tfrac{1}{2}x) - U_{k-1}(\tfrac{1}{2}x) = \{\sin(2k+2)\,\theta - \sin 2k\theta\}/\sin 2\theta$$
$$= \cos(2k+1)\,\theta/\cos\theta = T_{2k+1}(X)/X,$$

the equation $\quad T_n(X) = 0 \quad$ for B_n or $[3^{n-2}, 4]$ becomes

$$T_{\frac{1}{2}n}(\tfrac{1}{2}x) = 0 \qquad \text{when } n \text{ is even,}$$
$$U_{\frac{1}{2}(n-1)}(\tfrac{1}{2}x) - U_{\frac{1}{2}(n-3)}(\tfrac{1}{2}x) = 0 \qquad \text{when } n \text{ is odd.}$$

Since

$$T_k(\tfrac{1}{2}x) + T_{k-1}(\tfrac{1}{2}x) = \cos 2k\theta + \cos(2k-2)\,\theta = 2XT_{2k-1}(X),$$

the equation $\quad XT_{n-1}(X) = 0 \quad$ for D_n or $[3^{n-3,1,1}]$ becomes

$$T_{\frac{1}{2}(n-1)}(\tfrac{1}{2}x) = 0 \qquad \text{when } n \text{ is odd,}$$
$$T_{\frac{1}{2}n}(\tfrac{1}{2}x) + T_{\frac{1}{2}n-1}(\tfrac{1}{2}x) = 0 \qquad \text{when } n \text{ is even.}$$

Combining these results with 'Table 1' of [17, p. 770] where $Y = \sqrt{x+2}$, we can tabulate the equations with $n \leqq 9$ as follows.

Group	Equation for $2\cos\frac{2m_\nu\pi}{h}$	h	$m_1, m_2, \ldots, m_n$
$\mathfrak{D}_p = [p]$	$x - 2\cos\frac{2\pi}{p} = 0$	p	$1, p-1$
$A_3 = [3, 3]$	$x = 0$	4	1, 2, 3
$B_3 = [4, 3]$	$x - 1 = 0$	6	1, 3, 5
$H_3 = [5, 3]$	$x - \tau = 0$	10	1, 5, 9
$A_4 = [3^3]$	$x^2 + x - 1 = 0$	5	1, 2, 3, 4
$D_4 = [3^{1,1,1}]$	$x^2 + x - 2 = 0$	6	1, 3, 3, 5
$B_4 = [4, 3^2]$	$x^2 - 2 = 0$	8	1, 3, 5, 7
$F_4 = [3, 4, 3]$	$x^2 - 3 = 0$	12	1, 5, 7, 11
$H_4 = [5, 3, 3]$	$x^2 - \tau^{-1}x - \tau^2 = 0$	30	1, 11, 19, 29
$A_5 = [3^4]$	$x^2 - 1 = 0$	6	1, 2, 3, 4, 5
$D_5 = [3^{2,1,1}]$	$x^2 - 2 = 0$	8	1, 3, 4, 5, 7
$B_5 = [4, 3^3]$	$x^2 - x - 1 = 0$	10	1, 3, 5, 7, 9
$A_6 = [3^5]$	$x^3 + x^2 - 2x - 1 = 0$	7	1, 2, 3, 4, 5, 6
$D_6 = [3^{3,1,1}]$	$x^3 + x^2 - 3x - 2 = 0$	10	1, 3, 5, 5, 7, 9
$B_6 = [4, 3^4]$	$x^3 - 3x = 0$	12	1, 3, 5, 7, 9, 11
$E_6 = [3^{2,2,1}]$	$x^3 + x^2 - 3x - 3 = 0$	12	1, 4, 5, 7, 8, 11
$A_7 = [3^6]$	$x^3 - 2x = 0$	8	1, 2, 3, 4, 5, 6, 7
$D_7 = [3^{4,1,1}]$	$x^3 - 3x = 0$	12	1, 3, 5, 6, 7, 9, 11
$B_7 = [4, 3^5]$	$x^3 - x^2 - 2x + 1 = 0$	14	1, 3, 5, 7, 9, 11, 13
$E_7 = [3^{3,2,1}]$	$x^3 - 3x - 1 = 0$	18	1, 5, 7, 9, 11, 13, 17
$A_8 = [3^7]$	$x^4 + x^3 - 3x^2 - 2x + 1 = 0$	9	1, 2, 3, 4, 5, 6, 7, 8
$D_8 = [3^{5,1,1}]$	$x^4 + x^3 - 4x^2 - 3x + 2 = 0$	14	1, 3, 5, 7, 7, 9, 11, 13
$B_8 = [4, 3^6]$	$x^4 - 4x^2 + 2 = 0$	16	1, 3, 5, 7, 9, 11, 13, 15
$E_8 = [3^{4,2,1}]$	$x^4 + x^3 - 4x^2 - 4x + 1 = 0$	30	1, 7, 11, 13, 17, 19, 23, 29
$A_9 = [3^8]$	$x^4 - 3x^2 + 1 = 0$	10	1, 2, 3, 4, 5, 6, 7, 8, 9
$D_9 = [3^{6,1,1}]$	$x^4 - 4x^2 + 2 = 0$	16	1, 3, 5, 7, 8, 9, 11, 13, 15
$B_9 = [4, 3^7]$	$x^4 - x^3 - 3x^2 + 2x + 1 = 0$	18	1, 3, 5, 7, 9, 11, 13, 15, 17

Among the exponents m_ν for D_n, we notice that the rhythm of the sequence of odd numbers is broken by the number $\frac{1}{2}h=n-1$. When n is odd, it comes between $n-2$ and n. When n is even, it is repeated (since the polynomial $XT_{n-1}(X)$ is then divisible by X^2).

Comparing the equations for F_4, H_4, E_6 and E_8, we notice that

$$x^3+x^2-3x-3=(x^2-3)(x+1)$$

and

$$x^4+x^3-4x^2-4x+1=(x^2-\tau^{-1}x-\tau^2)(x^2+\tau x-\tau^{-2}),$$

confirming the possibility of projecting 2_{21} into $\{3, 4, 3\}$ plus a repeated point, and projecting 4_{21} into $\{3, 3, 5\}$ plus $\{3, 3, \frac{5}{2}\}$.

In 3_{21} and 4_{21}, as drawn in Fig. 3.7b and Fig. 3.8c, the symmetry operation $R_1 R_2 \ldots R_n$ appears as a rotation of period h (=18 or 30, respectively). For the analogous view of 2_{21}, see [15, p. 463].

As we saw in [21, pp. 225–226], the groups that contain the central inversion $(R_1 R_2 \ldots R_n)^{\frac{1}{2}h}$ are those for which h is even while every m_ν is odd and, if n is odd, $\frac{1}{2}h$ is odd too, namely

$$[p]\ (p\ \text{even}),\quad B_n,\quad D_n\ (n\ \text{even}),\quad H_3,\quad H_4,\quad F_4,\quad E_7\quad \text{and}\quad E_8.$$

In the last case, the central quotient group of the 'even' subgroup E_8^+ is the simple group of order $48\times 10! = 174182400$, namely

$$E_8^+/\mathfrak{S}_2 = O_8^+(2)$$

[9, p. 85].

Let us say that an isometry is of *type* ν if it is expressible as the product of ν (but no fewer) reflections. Such an isometry leaves invariant an $(n-\nu)$-flat: the intersection of the ν mirrors. Thus the identity is of type 0, a reflection is of type 1, and an isometry of type n fixes only one point. In any one of the irreducible reflection groups, let b_ν denote the number of isometries of type ν. Thus $b_0=1$, b_1 is the total *number of reflections* occurring in the group, and

$$b_0+b_1+\ldots+b_n=\Gamma,$$

the *order* of the group. According to a remarkable theorem of Shephard and Todd [52, pp. 279–283, 296–301; 53] the generating function for these numbers b_ν is the product of n linear factors:

$$b_0+b_1 t+b_2 t^2+\ldots+b_n t^n=(1+m_1 t)(1+m_2 t)\ldots(1+m_n t).$$

Thus the number of reflections is

$$b_1=m_1+m_2+\ldots+m_n=\tfrac{1}{2}nh$$

[6, p. 118; 17, p. 780; 21, p. 231; 22, p. 150] and the order of the group is

$$\Gamma=(m_1+1)(m_2+1)\ldots(m_n+1).$$

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Received October 21, 1987