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Regular and Semi-Regular Polytopes. II

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Contents

2.1. Introduction

During the years that have elapsed since the appearance of Part I in this *Zeitschrift* **46**, 380–407 (1940), some changes have taken place in notation and nomenclature. Accordingly it seems desirable to begin Part II by summarizing and amending Part I, which deals with polytopes and honeycombs in Euclidean 3-space, that is, with polyhedra and with space-filling collections of polyhedra (such as the cubic lattice $\delta_4=\{4, 3, 4\}$, regarded as a space-filling collection of congruent solid cubes $\gamma_3=\{4, 3\}$ such that every edge belongs to 4 of them).

Since we are not here concerned with 'star polytopes' or 'complex polytopes', we can follow Grünbaum [20, p. 31] by defining a *polytope* to be the convex hull of a finite set of points in Euclidean space. Those points of the set which cannot be removed without changing the hull are called *vertices*. The number of dimensions is variously denoted by d (by Grünbaum and his disciples) or l (as in our § 1.2). By now using the letter n, we are following Schläfli [26, p. 189] who, however, called a polytope a 'polyscheme'. (When $n=0$, 1, 2 or 3, an n-polytope is a point, line-segment, polygon or polyhedron.) For each k with $0\leq k\leq n$ there are a number of k-polytopes belonging to the n-polytope; these are called the N_k *k-faces*. In particular, the 0-faces are the vertices, the 1-faces are the *edges*, and the $(n-1)$-faces are the *facets* (or, in older literature, the *cells*). By declaring that there is one '(−1)-face' (the empty set) and one 'n-face' (the whole polytope), so that $N_{-1}=N_n=1$, we can express the famous Euler-Schläfli formula [11, p. 165] as

$$\sum_{k=-1}^{n} (-1)^k N_k = 0.$$

Among the various criteria for a *regular* polytope [20, p. 412], one of the simplest is the existence of a point (the *centre*) from which all the k-faces are at the same distance ${}_kR$, for $0 \leqq k < n$. For instance, a polygon is regular if it has a circumcircle and an incircle which are concentric. By allowing the circumradius ${}_0R$ to become infinite, we may regard an n-dimensional *honeycomb* (that is, a space-filling collection of n-polytopes) as a degenerate $(n+1)$-polytope whose 'facets' are the cells of the honeycomb. Then, of course, a *regular* honeycomb is one whose cells are regular and congruent; $N_k=\infty$ and ${}_kR=\infty$, for $0 \leqq k \leqq n$.

It follows from these definitions that the symmetry group of a regular polytope or honeycomb is transitive on its 'flags' [19, p. 63].

The definitions in § 1.1 and § 1.9 (p. 380, 400) can be extended recursively to any number of dimensions. A polytope or honeycomb is said to be *uniform* if it has the following two properties:

(i) each facet is a uniform polytope,
(ii) the symmetry groups is transitive on the vertices.

In § 1.3, the cosines of the six dihedral angles of a Euclidean tetrahedron were shown to be related by the determinantal Eq. (1.35). The 'elegant proof' was attributed to Thorold Gosset, but was in fact given much earlier by George Salmon [25, p. 48].

In § 1.2, we began to see that, in n dimensions, any finite group generated by reflections (or briefly, 'reflection group') can be generated by reflections $R_1, \ldots, R_n$ in n hyperplanes ('mirrors') forming an angular region (angle, trihedron, etc.) such that the dihedral angle between the μth and νth mirrors is $\pi/q_{\mu\nu}$, where $q_{\mu\nu}$ is a suitable integer greater than 1 (often equal to 2 or 3, and seldom greater than 5) [11, pp. 75–86, 187–196]. Moreover, the natural convention $q_{\mu\mu}=1$ enables us to present the group by the abstract definition

$$(R_\mu R_\nu)^{q_{\mu\nu}}=1 \qquad (\mu, \nu=1, \ldots, n). \tag{2.11}$$

Similarly, any infinite discrete reflection group can be generated by reflections in m mirrors $(n<m \leqq 2n)$ forming either a simplex or a 'prism': the Cartesian product (or 'rectangular product') of $m-n$ simplexes. In the latter case the group is a direct product.

Any reflection group G has an 'even' subgroup G^+, of index 2, generated by rotations which are products of pairs of generators of G. The elements of this rotation group are those elements of G whose expressions as 'words' $R_\alpha R_\beta \ldots$ are products of *even* numbers of R's. For instance, $[q]$ being the dihedral group $\mathfrak{D}_q$ of order $2q$, $[q]^+$ is the cyclic group $\mathfrak{C}_q$. This superscript + replaces the prime (′) which was unhappily used in § 1.4. In the revised notation, the 3-dimensional *point groups* (finite groups of isometries [9, pp. 274–277; 17, pp. 39, 135]) are:

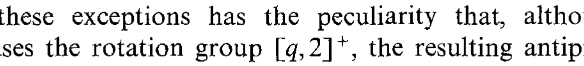
$[q]^+ \cong \mathfrak{C}_q$,	order q,	the cyclic rotation group;
$[q] \cong \mathfrak{D}_q$,	order $2q$,	the dihedral reflection group;
$[q,2]^+ \cong \mathfrak{D}_q$,	order $2q$,	the dihedral rotation group;
$[q^+,2] \cong \mathfrak{C}_q \times \mathfrak{D}_1$,	order $2q$,	cyclic with equatorial reflection;
$[2q^+,2^+] \cong \mathfrak{C}_{2q}$,	order $2q$,	generated by a rotatory inversion;
$[2q,2^+] \cong \mathfrak{D}_{2q}$,	order $4q$,	the group of the antiprism $s\begin{Bmatrix} q \\ 2 \end{Bmatrix}$;
$[q,2] \cong \mathfrak{D}_q \times \mathfrak{D}_1$,	order $4q$,	the group of the prism $\{q\} \times \{\ \}$;
$[3,3]^+ \cong \mathfrak{A}_4$,	order 12,	the tetrahedral (rotation) group;
$[3^+,4] \cong \mathfrak{A}_4 \times \mathfrak{S}_2$,	order 24,	the pyritohedral group (§ 1.6);
$[3,3] \cong \mathfrak{S}_4$,	order 24,	the group of the tetrahedron $\{3,3\}$;
$[3,4]^+ \cong \mathfrak{S}_4$,	order 24,	the octahedral group;
$[3,4] \cong \mathfrak{S}_4 \times \mathfrak{S}_2$,	order 48,	the group of the octahedron and cube;
$[3,5]^+ \cong \mathfrak{A}_5$,	order 60,	the icosahedral group;
$[3,5] \cong \mathfrak{A}_5 \times \mathfrak{S}_2$,	order 120,	the group of the icosahedron and dodecahedron.

Although both the groups $\mathfrak{D}_1$ and $\mathfrak{S}_2$ are isomorphic to $\mathfrak{C}_2$, we have used both symbols in this list to distinguish between the 'equatorial' reflection that belongs to $[q,2]$ and the 'central inversion' that belongs to $[3^+,4]$ and $[3,4]$ and $[3,5]$. (It is a happy coincidence that $\mathfrak{S}_2$ is Schoenflies's symbol for the group $[2^+,2^+]$ generated by the central inversion.)

We saw in § 1.5 that all the uniform polyhedra can be obtained by Wythoff's construction. This means that the set of vertices of each polyhedron is the orbit of a suitable point under the action of either a *reflection* group or a *rotation* group. (For an alternative treatment, see [12, pp. 16–18].) However, this happy state of affairs cannot be assumed to continue in higher spaces, although the number of known exceptions is remarkably small. Among the two-dimensional honeycombs, the only one unattainable by Wythoff's construction is the 'anomalous' tessellation $3^3 \cdot 4^2$, whose symmetry group is **cm** in the Hermann-Mauguin notation [17, pp. 44].

Near the end of § 1.5 (on p. 395), the graphical symbol for the octahedron $\{3,4\}$ accidentally lacks its central dot.

Among the uniform polyhedra, *reflection* groups suffice in every case except the antiprisms $s\begin{Bmatrix} q \\ 2 \end{Bmatrix}$, the snub cube $s\begin{Bmatrix} 4 \\ 3 \end{Bmatrix}$, and the snub dodecahedron $s\begin{Bmatrix} 5 \\ 3 \end{Bmatrix}$:



The first of these exceptions has the peculiarity that, although Wythoff's construction uses the rotation group $[q,2]^+$, the resulting antiprism has some planes of symmetry: its symmetry group $[2q,2^+]$ is neither a reflection group nor a rotation group, but a group generated by one reflection and one half-turn.

As we saw at the end of § 1.1, the snub cube and the snub dodecahedron are *chiral*. However, their vertex figures are pentagons, each having four congruent sides and thus having bilateral symmetry. An amusing consequence of this symmetry was noticed by Pieter Huybers [16]. The 24 vertices of the snub cube $s\begin{Bmatrix}4\\3\end{Bmatrix}$ have for coordinates the cyclic permutations of $(x^2, x, 1)$ with an even (or odd) number of minus signs, and the cyclic permutations of $(x, x^2, 1)$ with an odd (or even) number of minus signs, where x is the real root, about 0.54368903, of the cubic equation

$$x^3+x^2+x-1=0$$

[23, p. 176]. (The choice between 'even' and 'odd' distinguishes the *dextro* and *laevo* varieties of this chiral polyhedron.)

In § 1.9 we saw how Angelo Andreini's fourteen uniform honeycombs (in three dimensions) can be obtained by Wythoff's construction, although there are several anomalous honeycombs (analogous to the tessellation $3^3 \cdot 4^2$) which cannot be so obtained [5a, p. 184]. The empty space in the fourth column on p. 402 should have been used to define the symbol

$$h(\{\infty\} \times \{6,3\}) \quad \text{or} \quad h(\{6,3\} \times \{\infty\}),$$

analogous to $h\{6,3\}=\{3,6\}$ [11, p. 155]. In the third column on p. 403, the 'graphical symbols' for $h_3\delta_4$ (No. 19) and $h_{2,3}\delta_4$ (No. 23) were accidentally interchanged: the central dot should carry a ring in the case of $h_{2,3}\delta_4$, but not in the case of $h_3\delta_4$.

For the infinite groups $\triangle$, $\square$, $\begin{bmatrix}3\\3\\4\end{bmatrix}$, more convenient symbols are

$$[3^{[3]}], \quad [3^{[4]}], \quad [4, 3^{1,1}].$$

The Appendix (beginning on p. 404) deals with 'Cayley group-pictures', which are now known as *Cayley diagrams* (or 'Cayley graphs'). The table on p. 407 indicates that the groups

$$[4,3,4]^+, \quad \begin{bmatrix}3\\3\\4\end{bmatrix}^+ = [4,3^{1,1}]^+, \quad \square^+ = [3^{[4]}]^+, \quad [(4,3)^+,4]$$

have presentations (or 'abstract definitions') yielding Cayley graphs which consist of the vertices and edges of the honeycombs

$$t_{0,1,2}\delta_4, \quad t_{1,2}\delta_4, \quad h_2\delta_4, \quad t_{0,1,3}\delta_4.$$

In the second presentation for $[4,3,4]^+$ (yielding $t_{0,1,3}\delta_4$ again), E^3 was accidentally printed as E^2.

Concluding this Introduction, I would like to express my gratitude to Patrick Du Val, A.C. Hurley, N.W. Johnson and Joachim Neubüser for some helpful ideas and references.

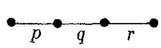
2.2. The Four-Dimensional Reflection Groups

A finite n-dimensional reflection group is naturally regarded as operating on the unit $(n-1)$-sphere whose centre is the common point of the mirrors for the n generating reflections $R_1, \ldots, R_n$. The mirrors intersect the $(n-1)$-sphere in n 'great $(n-2)$-spheres' forming a spherical simplex which has a dihedral angle $\pi/q_{\mu\nu}$ between the μth and νth facets [11, pp. 79–81, 188–190]. The graph for this *fundamental region* has n vertices or *dots*, representing the n facets of the simplex (or the n generators of the group), and two of the dots are joined by an edge or *link* whenever $q_{\mu\nu} > 2$ so that the corresponding facets form an acute angle (and the two generators R_μ and R_ν are non-commutative). The link is unmarked when $q_{\mu\nu} = 3$ (the most prevalent value) but carries the *mark* $q_{\mu\nu}$ whenever $q_{\mu\nu} > 3$. In particular, the μth and νth dots are joined by an unmarked link if $R_\mu R_\nu R_\mu = R_\nu R_\mu R_\nu$, but are not (directly) joined if $R_\mu R_\nu = R_\nu R_\mu$.

If no dot belongs to more than two links, the fundamental region is the special kind of simplex called an *orthoscheme* [26, p. 243]. In this case the facets (i.e., the mirrors) occur in a natural order such that any two which are not consecutive are perpendicular; that is, any two non-consecutive generators are commutative. Such a group is denoted by

$$[q_{12}, q_{23}, q_{34}, \ldots], \qquad \text{or simply } [p, q, r, \ldots].$$

In the 4-dimensional case it is found that all but one of the finite reflection groups are of this kind, namely

 $= [p, q, r] \quad (= [r, q, p])$

for suitable values of the integers p, q, r $(\geqq 2)$, with the presentation

$$\begin{aligned} R_1^2 = R_2^2 = R_3^2 = R_4^2 &= (R_1 R_3)^2 = (R_1 R_4)^2 = (R_2 R_4)^2 \\ &= (R_1 R_2)^p = (R_2 R_3)^q = (R_3 R_4)^r = 1, \end{aligned} \tag{2.21}$$

generalizing (1.45). It is to be understood that the mark (p or q or r) is omitted from the graph if its value is 3, and that the whole link is omitted if the value is 2. For instance, each 3-dimensional reflection group $[p, q]$ $(= [q, p])$, where $p^{-1} + q^{-1} > \frac{1}{2}$, yields a direct product

$$[p, q] \times \mathfrak{D}_1 = [p, q, 2] = [q, p, 2] = [2, p, q] = [2, q, p]$$

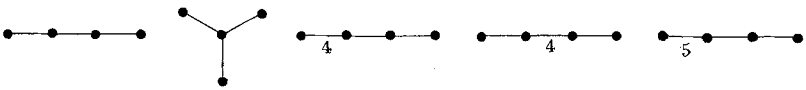
when we add an isolated dot to the graph for $[p, q]$. In this case the order of the group is twice that of $[p, q]$, and the fundamental region is a spherical tetrahedron in which the faces at one particular vertex form the trihedral fundamental region for $[p, q]$ while the fourth face is perpendicular to all those three.

In another direct product

$$[p, 2, r] = [p] \times [r] \cong \mathfrak{D}_p \times \mathfrak{D}_r,$$

the graph $\bullet\!\underset{p}{\text{—}}\!\bullet \quad \bullet\!\underset{r}{\text{—}}\!\bullet$ indicates a spherical tetrahedron having dihedral angles π/p and π/r along two opposite edges while the remaining four dihedral angles are $\pi/2$.

In 4 dimensions there remain just five 'irreducible' groups



$$[3,3,3], \qquad \begin{bmatrix}3\\3\\3\end{bmatrix} \qquad [4,3,3], \qquad [3,4,3], \qquad [5,3,3]$$

[11, pp. 196, 297] or, in the notation of Cartan and Bourbaki [2, pp. 293–194],

$$A_4, \qquad D_4, \qquad B_4, \qquad F_4, \qquad H_4.$$

Their orders

$$120, \qquad 192, \qquad 384, \qquad 1152, \qquad 14400$$

may be computed in various ways, as we shall soon see. The uncomfortably tall symbol for the second group is conveniently abbreviated to

$$[3^{1,1,1}].$$

Its fundamental region is a spherical pyramid based on an equilateral triangle. The dihedral angles along the edges of this base (represented by the three links in the Y-shaped graph) are all $\pi/3$, while the angles along the 'oblique' edges are right angles. In other words, this triangular pyramid $abcd$ is right-angled along the edges ad, bd, cd through its apex d, while the angles along the edges bc, ca, ab of its equilateral base (see Fig. 2.2a) are $\pi/3$: greater than these angles would be for the analogous pyramid in Euclidean space.

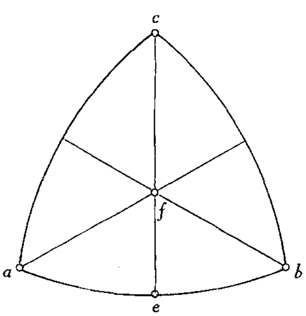


Fig. 2.2a. The Base of the Fundamental Region for $[3^{1,1,1}]$

If we split the pyramid into two halves by a 'vertical' plane joining the apex d to a median ce of the base, one half, $aedc$, having right angles along its edges ad, de, ec is an orthoscheme. Since the dihedral angles along its remaining edges dc, ca, ae are $\pi/4$, $\pi/3$, $\pi/3$, this orthoscheme is the fundamental region for [4, 3, 3]. If f is the centre of the base abc, the triangle aef, being one sixth of abc, is one third of the face aec of the orthoscheme $aedc$. Hence vertical planes joining d to the remaining medians of abc will decompose the pyramid $abcd$ into six replicas of the tetrahedron $aefd$ which (having right angles along its edges af, fe, ed and angles $\pi/3$, $\pi/4$, $\pi/3$ along fd, da, ae) is another orthoscheme, namely the fundamental region for [3, 4, 3]. Since three such orthoschemes will fit together to fill up $aedc$, the group [3, 4, 3] *contains* [4, 3, 3] *as a subgroup of index* 3. Also we have seen that [4, 3, 3] *contains* $[3^{1,1,1}]$ *as a subgroup of index* 2.

Since $[p, q, r]$ has the presentation (2.21), we can find its order by enumerating the cosets of the known subgroup $[p, q]$ generated by R_1, R_2, R_3 [17, pp. 12, 123]. A more amusing procedure is to obtain the order of [p, 3, 3] as $2\pi^2/V$, where V is the volume of the fundamental region. In terms of the Schläfli function

$$F_4(\alpha)=\frac{4}{\pi^2}\int_{\kappa}^{\alpha} y\,dx, \tag{2.22}$$

where $\sec 2y=\sec 2x-2$ and $\sec 2\kappa=3$, the volume of such a spherical orthoscheme is

$$V=\frac{\pi^2}{8}F_4\left(\frac{\pi}{p}\right)=\frac{1}{2}\int_{\kappa}^{\frac{\pi}{p}} y\,dx$$

[8, pp. 182–184]. We thus find the order of the group [p, 3, 3] to be

$$16/F_4\left(\frac{\pi}{p}\right). \tag{2.23}$$

It is remarkable that the exact value of this Schläfli function is known for eleven values of α:

$$\begin{array}{llllllllllll} \alpha=\kappa, & \pi/5, & \pi/4, & \pi/3, & 2\pi/5, & \pi/2, & 3\pi/5, & 2\pi/3, & 3\pi/4, & 4\pi/5, & \pi-\kappa; \\ F_4(\alpha)=0, & 1/900, & 1/24, & 2/15, & 191/900, & 1/3, & 409/900, & 8/15, & 5/8, & 599/900, & 2/3. \end{array}$$

From the fourth, third and second we see that the order of [p, 3, 3] is

$$120 \text{ for } p=3, \qquad 384 \text{ for } p=4, \qquad 14400 \text{ for } p=5;$$

consequently the orders of $[3^{1,1,1}]$ and [3, 4, 3] are $384/2=192$ and $384\times 3=1152$.

A third method [11, pp. 221, 232] expresses the order of $[p, q, r]$ (with p, q, r all greater than 2) in the form

$$64h\bigg/\left(12-p-2q-r+\frac{4}{p}+\frac{4}{r}\right), \tag{2.24}$$

where h (the period of $R_1 R_2 R_3 R_4$) is given by the quadratic equation

$$x^2-\left(\cos^2\frac{\pi}{p}+\cos^2\frac{\pi}{q}+\cos^2\frac{\pi}{r}\right)x+\cos^2\frac{\pi}{p}\cos^2\frac{\pi}{r}=0,$$

whose greater root is $\cos^2(\pi/h)$. This result incidentally provides an algebraic criterion for the finiteness of $[p, q, r]$ (with $p, q, r > 2$):

$$p + 2q + r - \frac{4}{p} - \frac{4}{q} < 12 \tag{2.25}$$

[10, pp. 8–10, 24–25].

When $p = r$, the orthoscheme $P_1 P_2 P_3 P_4$ is symmetrical by a half-turn T about the join of the midpoints of two edges: the longest edge $P_1 P_4$ and the opposite edge $P_2 P_3$. Adjoining to the reflection group $[p, q, p]$ this half-turn (which transforms each R_ν into $R_{5-\nu}$), we obtain a 'mixed' group $[[p, q, p]]$, generated by R_1, T, and R_3. Apart from the trivial case when $p = 2$, the finite groups of this kind are

$$[[p, 2, p]], \quad [[3, 3, 3]], \quad [[3, 4, 3]]$$

of order $8p^2$, 240, 2,304, analogous to the infinite group [[4, 3, 4]] which was described in § 1.4. For $[[p, q, p]]$, the presentation

$$R_1^2 = T^2 = R_3^2 = (R_1 T)^4 = (TR_3)^{2q} = (R_1 R_3)^2 = (R_1 TR_3 T)^p = 1 \tag{2.26}$$

[8, p. 91, (4.7) with T for R_2] shows that this is the complete symmetry group of the regular *skew* polyhedron $\{4, 2q \mid p\}$. When $q = 2$, the skew polyhedron is a geometric realization of the regular *map* $\{4, 4\}_{p,0}$ [17, pp. 104, 109], so

$$[[p, 2, p]] \cong [4, 4]_{p,0} \quad \text{(of order } 8p^2\text{)}. \tag{2.27}$$

When $p = 3$ (and $q = 2$ or 3 or 4), we can use, for $[[3, q, 3]]$, the *two* generators $A = R_1$ and $B = R_1 TR_3$, with the presentation

$$A^2 = (AB)^{2q} = (AB^{-1} AB)^2 = (AB^{-2} AB^3)^2 = 1 \tag{2.28}$$

[15, pp. 327, 331]. In fact, these relations imply

$$R_1 = A, \quad R_2 = B^2 AB^{-2}, \quad R_3 = B^{-1} AB^2 AB^{-2} AB, \quad R_4 = BAB^{-1},$$
$$T = B^2 AB^{-2} AB. \tag{2.29}$$

When $q = 2$, (2.28) provides an alternative presentation for $[[3, 2, 3]] \cong [4, 4]_{3,0}$.

In the case of [3, 4, 3], the generators R_1, R_2, R_3, R_4 are reflections in the face-planes *ecd*, *afd*, *abd*, *abc* of the orthoscheme *aefd* (see Fig. 2.2 A). Since the mirror *afd* reflects *abd* into *acd*, the reflection in *acd* is R_3 transformed by R_2, that is, $R_2 R_3 R_2$. Thus [3, 4, 3] has a subgroup [4, 3, 3] (with fundamental region *aedc*) generated by the four reflections

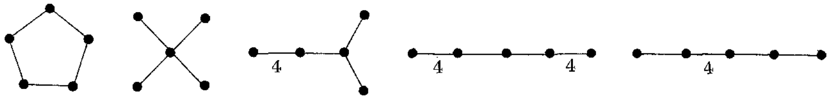
$$R_1, \quad R_2 R_3 R_2, \quad R_4, \quad R_3.$$

Other subgroups of [3, 4, 3] and [[3, 4, 3]] are described elsewhere [15, pp. 314–328]. (Unhappily, in the title for § 12, the order of [[3, 4, 3]] was printed as 1,152 instead of 2,304.)

As we saw on page 564, there are five finite irreducible reflection groups

$$[3,3,3]=A_4,\quad [3^{1,1,1}]=D_4,\quad [4,3,3]=B_4,\quad [3,4,3]=F_4,\quad [5,3,3]=H_4.$$

It is interesting to observe that there are also five infinite groups



$$[3^{[5]}]=\tilde{A}_4,\quad [3^{1,1,1,1}]=\tilde{D}_4,\quad [4,3,3^{1,1}]=\tilde{B}_4,\quad [4,3,3,4]=\tilde{C}_4,\quad [3,4,3,3]=\tilde{F}_4,$$

whose fundamental regions are *Euclidean* 4-simplexes [2, p. 199; 7, p. 412]. The four mirrors at any vertex P of such a simplex generate a finite subgroup whose symbol is derived by omitting one of the five dots (and any incident links). P is called a *special* vertex [11, pp. 191, 205] if this subgroup is maximal; then we call it a *special* subgroup. Thus the special subgroups of $\tilde{A}_4$, $\tilde{D}_4$, $\tilde{B}_4$, $\tilde{C}_4$, $\tilde{F}_4$ are A_4, D_4, B_4, B_4 again and F_4. The simplex has f special vertices where f is 5 for $\tilde{A}_4$ (as all 5 are special), 4 for $\tilde{D}_4$, 2 for $\tilde{B}_4$ and $\tilde{C}_4$, and 1 for $\tilde{F}_4$ (indicated by the 'last' dot). The fundamental region for $\tilde{D}_4$ or $\tilde{B}_4$ is decomposed into 2 or 3 pieces by 1 or 2 hyperplanes through a special vertex. These are the same hyperplanes that decompose the fundamental regions for $D_4=[3^{1,1,1}]$ and $B_4=[4,3,3]$ in the manner already described. Since both $\tilde{B}_4$ and $\tilde{C}_4$ have B_4 at either of their 2 special vertices, while $\tilde{D}_4$ has D_4 at any of its 4 special vertices, and since the spherical orthoscheme $aedc$ for B_4 is one half of the pyramid $abcd$ for D_4, the simplex for $\tilde{C}_4$ is one half of the simplex for $\tilde{B}_4$, and this in turn is one half of the simplex for $\tilde{D}_4$. Finally, since the orthoscheme $aefd$ for F_4 is one third of the orthoscheme $aedc$ for B_4, the orthoscheme for $\tilde{F}_4$ is one third of the orthoscheme for $\tilde{B}_4$. It follows that $\tilde{D}_4$ is a subgroup of index 2 in $\tilde{B}_4$ while $\tilde{B}_4$ is of index 2 in $\tilde{C}_4$ and of index 3 in $\tilde{F}_4$. Since $\tilde{F}_4=[3,4,3,3]$ satisfies the relations

$$\mathrm{R}_\nu^2=(\mathrm{R}_1\mathrm{R}_2)^3=(\mathrm{R}_2\mathrm{R}_3)^4=(\mathrm{R}_3\mathrm{R}_4)^3=(\mathrm{R}_4\mathrm{R}_5)^3=1$$

while all other pairs of the five R_ν commute, we can verify the index of $\tilde{B}_4$ in $\tilde{F}_4$ by counting the cosets of the subgroup $\tilde{B}_4$ generated by

$$\mathrm{R}_1,\quad \mathrm{R}_2\mathrm{R}_3\mathrm{R}_2,\quad \mathrm{R}_4,\quad \begin{matrix}\mathrm{R}_3,\\ \mathrm{R}_5.\end{matrix}$$

(Notice that the product $\mathrm{R}_1\cdot\mathrm{R}_2\mathrm{R}_3\mathrm{R}_2=(\mathrm{R}_2\mathrm{R}_1)^2\mathrm{R}_3\mathrm{R}_2=\mathrm{R}_2\mathrm{R}_1\cdot\mathrm{R}_2\mathrm{R}_3\cdot\mathrm{R}_1\mathrm{R}_2$ has the same period as $\mathrm{R}_2\mathrm{R}_3$.)

2.3. Some Subgroups of Small Index

Each reflection group $[p,q,r]$ has its rotation subgroup $[p,q,r]^+$, of index 2, generated by $\mathrm{R}_1\mathrm{R}_2$, $\mathrm{R}_2\mathrm{R}_3$, $\mathrm{R}_3\mathrm{R}_4$ [10, pp. 25–26]. In particular,

$$[p,q,2]^+\cong[p,q], \tag{2.31}$$

but now the generators R_1, R_2, R_3 of $[p,q]$, instead of being reflections in planes of 3-space, are half-turns about the same planes embedded in 4-space. The remaining finite cases are

$$[p,2,r]^+,\ [3,3,3]^+,\ [4,3,3]^+,\ [3,4,3]^+,\ [5,3,3]^+,$$

of order $2pr$, 60, 192, 576, 7200. When p and r are relatively prime, $[p,2,r]^+ \cong \mathfrak{D}_{pr}$.

Also $[3^{1,1,1}]$ has its rotation subgroup $[3^{1,1,1}]^+$, of order 96.

In the case of $[[3,q,3]]$, generated by R_1, T, R_3 as in (2.26), the rotation subgroup $[[3,q,3]]^+$, generated by $R_1R_4=(R_1T)^2$, T and R_1R_3, is more simply generated by

$$B=R_1TR_3 \quad \text{and} \quad C=R_1R_3T,$$

in terms of which $(R_1T)^2=BC$, $T=B^2C^2B$, $R_1R_3=CB^2C^2B$ and the presentation is

$$(BC)^2=(B^3C^2)^2=(B^2C^3)^2=(B^{-1}C)^q=1 \tag{2.23}$$

[15, p. 326 (11.2)].

There is no such elegant presentation for the remaining rotation groups $[[p,q,p]]^+$; but another subgroup of index 2 in $[[p,q,p]]$ is

$$[[p,q,p]^+],$$

generated by $R=R_1T$ and $S=TR_3$, with the presentation

$$R^4=S^{2q}=(RS)^2=(RS^{-1})^p=1, \tag{2.33}$$

which shows that

$$[[p,q,p]^+] \cong (4,2q\,|\,2,p). \tag{2.34}$$

This is the rotation group of the regular skew polyhedron $\{4,2q\,|\,p\}$ [8, p. 92; 17, p. 109]. The finite cases are

$$[[p,2,p]^+],\ [[3,3,3]^+],\ [[3,4,3]^+],$$

of order $4p^2$, 120, 1152. For the last of these, see also Warner [30, p. 386]. The infinite group $[[4,3,4]^+] \cong (4,6\,|\,2,4)$ was mentioned in § 1.4.

When q is odd, the group $[p,q,r]^+$ is generated by the two rotations R_1R_2 and R_3R_4, without the help of R_2R_3. For then R_2R_3 is a power of

$$(R_2R_3)^2=(R_1R_2)^{-1}R_3R_4\cdot R_1R_2(R_3R_4)^{-1}.$$

On the other hand, when q is even, $[p,q,r]^+$ has a subgroup of index 2, say

$$[p^+,q,r^+],$$

generated by $Q=R_1R_2$ and $S=R_3R_4$, with the presentation

$$Q^p = S^r = (Q^{-1} S^{-1} QS)^{q/2} = 1 \tag{2.35}$$

[28, p. 76]. The finite cases are

$$[p^+, 2, r^+] \cong \mathfrak{C}_p \times \mathfrak{C}_r \quad \text{and} \quad [3^+, 4, 3^+],$$

of order pr and 288 [15, p. 314].

When $p = r$, we can adjoin T, as in (2.26), to obtain $[[p^+, q, p^+]]$ (q even), generated by Q and T:

$$Q^p = T^2 = \{(Q^{-1} T)^2 (QT)^2\}^{q/2}. \tag{2.36}$$

In particular,

$$[[p^+, 2, p^+]] \cong p[4]2, \tag{2.37}$$

of order $2p^2$ [17, p. 77], which is $\mathfrak{D}_p \times \mathfrak{C}_p$ when p is odd.

Three other subgroups of index 2 in $[p, q, r]$ are:

$[p^+, q, r]$	(q even),	generated by $R_1 R_2$, R_3, R_4,
$[(p, q)^+, r]$	(r even),	generated by $R_1 R_2$, $R_2 R_3$, R_4,
$[p, q^+, r]$	(p and r even),	generated by R_1, $R_2 R_3$, R_4.

The finite cases are:

$[2^+, 4, 3] \cong [4, 3]$, $\quad [3^+, 4, 2] \cong [4, 3^+, 2] \cong [3^+, 4] \times [1]$,
$[p^+, 2, r] \cong [p]^+ \times [r]$, $\quad [3^+, 4, 3]$ (of order 576),
$[(p, 2)^+, r]$ (of order $2pr$), $\quad [(p, q)^+, 2] \cong [p, q]^+ \times [1]$,
$[p, 2^+, r]$ (of order $2pr$), $\quad [(3, 3)^+, 4]$ (of order 192).

It is interesting to observe that, if p and r are both odd, the direct product $[p] \times [r] \cong [p, 2, r]$ is generated by the two elements

$$U = R_1 R_2 R_4, \quad V = R_1 R_3 R_4,$$

in terms of which

$$R_4 = U^p, \quad R_1 = V^r, \quad R_2 = V^r U^{p+1}, \quad R_3 = V^{r+1} U^p.$$

The presentation

$$U^{2p} = V^{2r} = (UV)^2 = (UV^{-1})^2 = 1$$

shows that

$$[p, 2, r] \cong (2p, 2r \mid 2, 2) \quad (p \text{ and } r \text{ odd}). \tag{2.38}$$

(Coxeter and Moser [17, p. 110] failed to notice that, when p and r are odd, $(2p, 2r \mid 2, 2) \cong \mathfrak{D}_p \times \mathfrak{D}_r$.)

If r is even while p remains odd, we have $V^r = 1$. Since $[(p, 2)^+, r]$, of order $2pr$, is generated by the three elements $R_1 R_2 = U^{p+1}$, $R_1 R_3 = VU^p$, $R_4 = U^p$, we have

$$[(p, 2)^+, r] \cong (2p, r \mid 2, 2) \quad (p \text{ odd}, r \text{ even}).$$

If p and r are both even, so that $U^p = V^r = 1$, the group $(p, r \mid 2, 2)$ (of order pr, generated by $U = R_1 R_2 R_4$ and $V = R_1 R_3 R_4$) is a subgroup of index 2 in $[(p, 2)^+, r]$ (p and r even).

The same abstract group $(p, r \mid 2, 2)$, defined by

$$U^p = V^r = (UV)^2 = (UV^{-1})^2 = 1 \qquad (p \text{ and } r \text{ even}),$$

arises again as a subgroup of index 2 in $[p, 2^+, r]$ (generated by R_1, $R_2 R_3$, R_4); but now, instead of $R_1 R_2 R_4$ and $R_1 R_3 R_4$, the generators are

$$U = R_1 R_2 R_3, \qquad V = R_2 R_3 R_4.$$

When $p = r$, we can augment the group $[p, 2^+, p]$ (generated by R_1, $S = R_2 R_3$, and R_4) by adjoining T (which transforms R_4 into R_1 and commutes with S) so as to obtain a group of order $4p^2$ with the presentation

$$R_1^2 = S^2 = T^2 = (R_1 S)^p = (ST)^2 = (R_1 T)^4 = (R_1 ST)^4 = 1.$$

In terms of $A = SR_1$, $B = R_1 T$, $C = TR_1 S$, this becomes

$$A^p = B^4 = C^4 = (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = 1$$

[17, pp. 96, 139]. Thus

$$[[p, 2^+, p]] \cong G^{4,4,p} \qquad (p \text{ even}). \tag{2.39}$$

As a limiting case we may allow p and r to become infinite while we still have $q = 2$, so that the orthoscheme in spherical space becomes an infinitely tall prism whose cross-section is a rectangle (in Euclidean space). We may then work in the plane of this cross-section, and the groups related to $[\infty, 2, \infty]$ reproduce nine of Federov's seventeed 2-dimensional space groups [17, pp. 40–47]:

$$\mathbf{pmm} \cong [\infty, 2, \infty] \cong [[\infty, 2, \infty]]^+,$$
$$\mathbf{p2} \cong [\infty, 2, \infty]^+, \quad \mathbf{p1} \cong [\infty^+, 2, \infty^+],$$
$$\mathbf{pm} \cong [\infty^+, 2, \infty] \cong [[\infty^+, 2, \infty^+]],$$
$$\mathbf{cmm} \cong [\infty, 2^+, \infty], \quad \mathbf{pmg} \cong [(\infty, 2)^+, \infty],$$
$$\mathbf{p4m} \cong [[\infty, 2, \infty]] \cong [[\infty, 2^+, \infty]],$$
$$\mathbf{p4} \cong [[\infty, 2, \infty]^+], \quad \mathbf{pgg} \cong (\infty, \infty \mid 2, 2).$$

It is natural to ask how the above results can be reconciled with the published lists of crystallographic and non-crystallographic point groups. In the accompanying table, the second column gives the symbol of Du Val [19, pp. 57, 61], which was used also by Warner [30, p. 386]. The third column gives the symbol of Brown et al. [3, pp. 360, 369–376, 386–390, 397–403], which begins with the order and continues, in each crystallographic case, with an arbitrary serial number; '(non-c.)' stands for 'non-crystallographic'. The last column gives, in some cases, an alternative symbol for the abstract group or 'isomorphism type': usually the symbol as a permutation group. For a detailed account of all the crystallographic point groups, see Hurley [22].

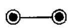
Presentation symbol	Du Val's symbol	Order and serial number	Alternative symbol
$[3,3,2]^+$	$(\mathbf{T}/\mathbf{C}_2;\mathbf{T}/\mathbf{C}_2)$	24.10	$\mathfrak{C}_2\times\mathfrak{A}_4$
$[(3,3)^+,2]$	$(\mathbf{T}/\mathbf{C}_1;\mathbf{T}/\mathbf{C}_1)^*_c$		
$[[3,2,3]^+]$	$(\mathbf{D}_3/\mathbf{C}_3;\ \mathbf{D}_3/\mathbf{C}_3)^*$	36.14	$(4,4\mid 2,3)$
$[4,3,2]^+$	$(\mathbf{O}/\mathbf{C}_2;\mathbf{O}/\mathbf{C}_2)$		
$[3,3,2]$	$(\mathbf{O}/\mathbf{C}_1;\mathbf{O}/\mathbf{C}_1)^*$	48.36	$\mathfrak{C}_2\times\mathfrak{S}_4$
$[(4,3)^+,2]$	$(\mathbf{O}/\mathbf{C}_1;\mathbf{O}/\mathbf{C}_1)^*_-$		
$[3^+,4,2]$	$(\mathbf{T}/\mathbf{C}_2;\mathbf{T}/\mathbf{C}_2)^*_c$	48.22	$\mathfrak{D}_2\times\mathfrak{A}_4$
$[3,3,3]^+$	$(\mathbf{I}^\dagger/\mathbf{C}_1;\mathbf{I}/\mathbf{C}_1)^\dagger$	60.13	$\mathfrak{A}_5$
$[3^{1,1,1}]^+$	$(\mathbf{T}/\mathbf{V};\mathbf{T}/\mathbf{V})$	96.1	
$[4,3,2]$	$(\mathbf{O}/\mathbf{C}_2;\mathbf{O}/\mathbf{C}_2)^*$	96.5	$\mathfrak{D}_2\times\mathfrak{S}_4$
$[3,3,3]$	$(\mathbf{I}^\dagger/\mathbf{C}_1;\mathbf{I}/\mathbf{C}_1)^{\dagger*}$	120.1	$\mathfrak{S}_5\cong(4,6\mid 2,3)$
$[[3,3,3]^+]$	$(\mathbf{I}^\dagger/\mathbf{C}_1;\mathbf{I}/\mathbf{C}_1)^{\dagger*}_-$		
$[[3,3,3]]^+$	$(\mathbf{I}^\dagger/\mathbf{C}_2;\mathbf{I}/\mathbf{C}_2)^\dagger$	120.2	
$[5,3,2]^+$	$(\mathbf{I}/\mathbf{C}_2;\mathbf{I}/\mathbf{C}_2)$	120 (non-c.)	$\mathfrak{C}_2\times\mathfrak{A}_5$
$[(5,3)^+,2]$	$(\mathbf{I}/\mathbf{C}_1;\mathbf{I}/\mathbf{C}_1)^*$		
$[4,3,3]^+$	$(\mathbf{O}/\mathbf{V};\mathbf{O}/\mathbf{V})$	192.3	
$[(3,3)^+,4]$	$(\mathbf{T}/\mathbf{V};\mathbf{T}/\mathbf{V})^*$	192.1	
$[3^{1,1,1}]$	$(\mathbf{T}/\mathbf{V};\mathbf{T}/\mathbf{V})^*_-$	192.2	
$[[3,3,3]]$	$(\mathbf{I}^\dagger/\mathbf{C}_2;\mathbf{I}/\mathbf{C}_2)^{\dagger*}$	240.1	$\mathfrak{C}_2\times\mathfrak{S}_5$
$[5,3,2]$	$(\mathbf{I}/\mathbf{C}_2;\mathbf{I}/\mathbf{C}_2)^*$	240 (non-c.)	$\mathfrak{D}_2\times\mathfrak{A}_5$
$[3^+,4,3^+]$	$(\mathbf{T}/\mathbf{T};\mathbf{T}/\mathbf{T})$	288.1	
$[4,3,3]$	$(\mathbf{O}/\mathbf{V};\mathbf{O}/\mathbf{V})^*$	384.1	$\mathfrak{C}_2\wr\mathfrak{S}_4$
$[3,4,3]^+$	$(\mathbf{O}/\mathbf{T};\mathbf{O}/\mathbf{T})$	576.2	
$[3^+,4,3]$	$(\mathbf{T}/\mathbf{T};\mathbf{T}/\mathbf{T})^*$	576.1	
$[[3^+,4,3^+]]$	$(\mathbf{T}/\mathbf{T};\mathbf{O}/\mathbf{O})$	576 (non-c.)	
$[3,4,3]$	$(\mathbf{O}/\mathbf{T};\mathbf{O}/\mathbf{T})^*$	1152.1	
$[[3,4,3]^+]$	$(\mathbf{O}/\mathbf{T};\mathbf{O}/\mathbf{T})^*_-$	1152 (non-c.)	$(4,8\mid 2,3)$
$[[3,4,3]]^+$	$(\mathbf{O}/\mathbf{O};\mathbf{O}/\mathbf{O})$	1152 (non-c.)	
$[[3,4,3]]$	$(\mathbf{O}/\mathbf{O};\mathbf{O}/\mathbf{O})^*$	2304 (non-c.)	
$[5,3,3]^+$	$(\mathbf{I}/\mathbf{I};\mathbf{I}/\mathbf{I})$	7200 (non-c.)	
$[5,3,3]$	$(\mathbf{I}/\mathbf{I};\mathbf{I}/\mathbf{I})^*$	14400 (non-c.)	

2.4. Wythoff's Construction and Its Numerical Consequences

From a 4-dimensional kaleidoscope we can derive a uniform polytope whose vertices are the images of a point on the line of intersection of three of the four mirrors, or on the plane of intersection of two mirrors (equidistant from the remaining two), or on one mirror (equidistant from the remaining three), or quite inside the kaleidoscope (equidistant from all four). In other words, when the fundamental region for the reflection group is a spherical tetrahedron, a typical vertex of the uniform polytope is at a vertex of the tetrahedron, or on an edge (where it would be cut by the internal bisector of the dihedral angle at the opposite edge), or an a face (equidistant from the remaining faces), or quite inside (at the centre of the inscribed sphere). A suitable graphical symbol for the polytope is derived from the graph symbolizing the reflection group G (or its fundamental region) by drawing a ring round one, two, three, or all four dots, to distinguish those mirrors on which the chosen point does *not* lie. (This is the same procedure that was used in §1.9 to derive a uniform *honeycomb* from a *Euclidean* tetrahedron.)

By removing one of the four dots, we derive a subgraph which is the symbol for a facet or face or edge or vertex, sometimes accompanied by a disconnected piece of graph (or possibly several such pieces) representing a group which leaves that element totally invariant. The disconnected piece (or pieces) must be included when we wish to compute the number of such s-dimensional elements π_s in the polytope; for the whole subgraph, regardless of rings, symbolizes the maximal subgroup of G that preserves one π_s, and the *number* of π_s's is the *index* of this subgroup: there is a coset for each π_s. In other words, if Γ and γ are the orders of the groups G and g symbolized by the whole graph and subgraph, regardless of rings, the number of π_s's is Γ/γ [8, p. 43]. By removing further dots (adjacent to ringed dots) we eventually obtain symbols for *all* the elements of the polytope, and their number is always given by such a quotient Γ/γ, possibly doubled or tripled.

These ideas extend easily from 4 to n dimensions [8, p. 49; 11, p. 202]. For instance, when $n=2$, alternate edges π_1 of the hexagon

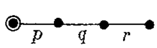


are represented by the two ringed dots, and although in this case $\Gamma/\gamma=6/2=3$, the number of edges is 6: three of each 'type'. When *all* the dots are ringed (as in this 2-dimensional example), the symbol for a *vertex* is the null graph, $\gamma=1$, and the polytope has Γ vertices. In other words, the vertices and edges of the uniform polytope constitute a Cayley diagram for G.

When $n-1$ of the n dots are ringed, we have a Cayley diagram for the rotation group G^+, as in the Appendix to Part I (see footnote 37 on p. 401).

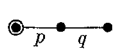
When the symbol for an element π_s is disconnected, so that the subgroup g is a direct product, it may happen that ringed dots occur in several pieces of the symbol. In the case π_s is the Cartesian (or 'rectangular') product of the elements symbolized by these pieces. For instance, two isolated dots, both ringed, symbolize a square, as on page 394 of Part I.

When only one dot is ringed, so that we are considering the images of one vertex P of the fundamental region, we can obtain a symbol for the *vertex figure* (whose vertices lie on the edges through P) by removing the one dot and ringing all neighboring dots. For instance, the *regular* polytope



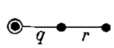
$=\{p,q,r\}$,

whose cell is



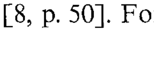
$=\{p,q\}$,

has vertex figure



$=\{q,r\}$.

A slight complication arises when the links joining the 'one dot' to its neighbours bear different marks [8, p. 50]. For instance, the vertex figure of



$=\mathrm{t}_1\{p,q,r\}$

is a prism of height $2\cos\pi/p$ whose base is an r-gon of edge $2\cos\pi/q$.

The notation $t_1 \Pi_n$ extends naturally to

$$t_{a,b,\ldots} \Pi_n$$

in which the rings are placed on the dots respresenting $R_{a+1}, R_{b+1}, \ldots$. In particular, $t_0 \Pi_n$ is Π_n itself, and $t_{n-1} \Pi_n$ is the reciprocal of Π_n. However, when two or more dots are ringed, there is no such simple rule to determine the vertex figure.

The graphical symbol for a uniform polytope (or honeycomb) also enables us to compute the number of r-dimensional elements π_r to which a given element π_s belongs (so that $s<r$). If there are any such elements π_r, any ringed pieces of the subgraph symbolizing π_s must, of course, occur in the subgraph symbolizing π_r. The complete symbol for π_s, regardless of rings, represents the subgroup g of G that preserves π_s. The common part of such symbols for π_r and π_s represents the subgroup g' of g that preserves not only π_s but also one of the incident π_r's. There is one such π_r for each coset of g' in g. Therefore the number of π_r's incident with π_s is equal to the index of g' in g. In particular, if the subgraph symbolizing π_r contains the whole of the subgraph symbolizing π_s, including any pieces that carry no ring, then $g'=g$ and the number is 1. If the two subgraphs have nothing in common, g' is trivial and the number is the order of g.

2.5. Four-Dimensional Polytopes

In dealing with the group $A_n \cong [3,3,\ldots]$ and the regular simplex $\alpha_n = \{3,3,\ldots\}$, it is natural to discard the familiar n-dimensional coordinates $(x_1,\ldots,x_n)$ in favour of $(n+1)$-dimensional coordinates $(u_1,\ldots,u_n,u_{n+1})$ with $\Sigma u_\nu = 0$, that is, to work in the hyperplane $\Sigma u_\nu = 0$ of a Euclidean $(n+1)$-space. (It is almost as appropriate to work in the parallel hyperplane $\Sigma u_\nu = 1$, in which case we could regard the u_ν as 'barycentric' coordinates.) The group $[3,3,\ldots] \cong \mathfrak{S}_{n+1}$ is generated by reflections in the n mirrors

$$u_1 = u_2, \quad u_2 = u_3, \ldots, \quad u_n = u_{n+1},$$

that is, by the consecutive transpositions

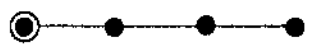
$$R_1 = (1\ 2), \quad R_2 = (2\ 3), \ldots, \quad R_n = (n\ \ n+1).$$

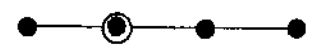
A typical point lying on all these mirrors except $u_{a+1} = u_{a+2}$ satisfies the remaining $n-1$ equations, so it has $a+1$ coordinates $n-a$ followed by $n-a$ coordinates $-(a+1)$. The $\binom{n+1}{a+1}$ vertices of $t_a \alpha_n$ are given by all the permutations of these $n+1$ coordinates. Similarly, a typical vertex of $t_{a,b,\ldots} \alpha_n$ is equidistant from the mirrors

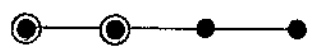
$$u_{a+1} = u_{a+2}, \quad u_{b+1} = u_{b+2}, \ldots$$

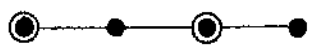
and lies on all the rest of them.

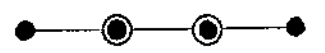
When $n=4$, these typical vertices are as follows:

for $\alpha_4 = t_0\,\alpha_4 =$  , $(4,\ -1,\ -1,\ -1,\ -1)$;

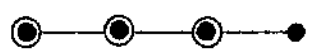
for $t_1\,\alpha_4 =$  , $(3,\ 3,\ -2,\ -2,\ -2)$;

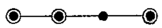
for $t_{0,1}\,\alpha_4 =$  , $(7,\ 2,\ -3,\ -3,\ -3)$;

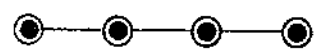
for $t_{0,2}\,\alpha_4 =$  , $(6,\ 1,\ 1,\ -4,\ -4)$;

for $t_{1,2}\,\alpha_4 =$  , $(1,\ 1,\ 0,\ -1,\ -1)$;

for $e\alpha_4 = t_{0,3}\,\alpha_4 =$  , $(1,\ 0,\ 0,\ 0,\ -1)$;

for $t_{0,1,2}\,\alpha_4 =$  , $(9,\ 4,\ -1,\ -6,\ -6)$;

for $t_{0,1,3}\,\alpha_4 =$  , $(8,\ 3,\ -2,\ -2,\ -7)$;

for $t_{0,1,2,3}\,\alpha_4 =$  , $(2,\ 1,\ 0,\ -1,\ -2)$

[14, p. 127]. The last of these, whose 120 vertices are given by all the permutations of 5 consecutive integers, is 'Hinton's polytope' [8, p. 73]. Berge [1, p. 135] calls it the 4-dimensional *permutohedron* and provides (on the next page) a good drawing of the analogous truncated octahedron $t_{0,1,2}\,\alpha_3 = t_{0,1}\,\beta_3$. For this polytope $t_{0,1,2,3}\,\alpha_4$, and also $t_{0,3}\,\alpha_4$, $t_{1,2}\,\alpha_4$, the complete symmetry group is not merely $[3,3,3] \cong \mathfrak{S}_5$ but the extension $[[3,3,3]]$. This includes the half-turn T that rotates the fundamental region for $[3,3,3]$ into itself, as we saw in §2.2. Since the symmetric group includes the half-turn (1 5)(2 4), whose product with T is the central inversion (transforming the simplex α_4 into its reciprocal $t_3\,\alpha_4$), we have

$$[[3,3,3]] \cong \mathfrak{S}_5 \times \mathfrak{C}_2.$$

There are, in general, $2^4 - 1 = 15$ ways to place rings on the 4 dots in the graph for $[p,q,r]$. But when $p=r$, as in the case of $[3,3,3]$ or $[3,4,3]$, the graph has bilateral symmetry and only nine of the fifteen polytopes are distinct: we have $t_3\,\alpha_4 = \alpha_4$, $t_2\,\alpha_4 = t_1\,\alpha_4$, and so on.

Ordinary n-dimensional coordinates are appropriate for the n-cube $\gamma_n = \{4,3,3,\ldots\}$ and its reciprocal $\beta_n = \{3,\ldots,3,4\}$, because their symmetry group $B_n \cong [4,3,3,\ldots]$, of order $2^n n!$, includes *all sign changes and permutations of the n coordinates*. In other words, $[4,3,3,\ldots]$ is the *wreath product* $\mathfrak{C}_2 \wr \mathfrak{S}_n$ [12, pp. 31, 118]. Its n generators are R_1, which reverses the sign of x_1, and the consecutive transpositions

$$R_2 = (1\ 2), \qquad R_3 = (2\ 3), \ldots, \qquad R_n = (n-1\ \ n).$$

They are represented by reflections in the hyperplanes

$$x_1 = 0, \qquad x_1 = x_2, \qquad x_2 = x_3, \ldots, \qquad x_{n-1} = x_n.$$

Solving $n-1$ of these n equations, we see that, if $a+b=n$, a typical vertex of $t_a\,\gamma_n = t_{b-1}\,\beta_n$ has, for coordinates, a zeros and b ones. The remaining vertices are obtained by changing the ones to ± 1. In particular, when $a=0$ and $n=4$, we see that the 16 vertices of the 4-cube or 'tesseract' γ_4 are

$$(\pm 1,\ \pm 1,\ \pm 1,\ \pm 1);$$

when $b=1$, that the 8 vertices of its reciprocal β_4 (the '16-cell') are $(\pm 1, 0, 0, 0)$ permuted; and when $b=2$, that the 24 vertices of the truncation $t_1\beta_4$ are $(\pm 1, \pm 1, 0, 0)$ permuted [11, p. 156].

Other polytopes related to β_4 and γ_4 are derived almost as easily. For instance, a typical vertex of

$$t_{0,1,2}\gamma_4 = t_{1,2,3}\beta_4$$

lies on the hyperplane $x_3 = x_4$ in a position equidistant from $x_1 = 0$, $x_1 = x_2$ and $x_2 = x_3$. Since the interior of the fundamental region is given by

$$x_1 > 0, \ x_1 < x_2 < x_3 < x_4,$$

we solve the equations

$$x_1 = \frac{x_2 - x_1}{\sqrt{2}} = \frac{x_3 - x_2}{\sqrt{2}}, \qquad x_3 = x_4$$

(with $x_1 = 1$ for convenience), obtaining

$$x_1 = 1, \ x_2 = \sqrt{2}+1, \qquad x_3 = x_4 = 2\sqrt{2}+1.$$

Thus the vertices of $t_{0,1,2}\gamma_4$ are the permutations of

$$(1, \sqrt{2}+1, 2\sqrt{2}+1, 2\sqrt{2}+1)$$

with arbitrary changes of sign [14, p. 130].

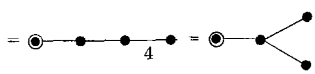
The coordinates $(\pm 1, \pm 1, 0, 0)$ (permuted) for $t_1\beta_4$ illustrate the role of the subgroup relationships that occur among the three groups

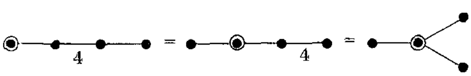
$$[3, 4, 3], \ [4, 3, 3], \ [3^{1,1,1}]$$

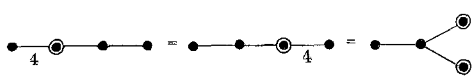
(see §2.2). In fact, along with the eight genuinely 'cubic' polytopes

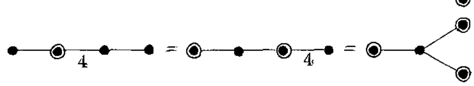
$$\gamma_4, \ t_{0,1}\gamma_4, \ t_{0,2}\gamma_4, \ t_{0,1,2}\gamma_4, \ t_{0,1,3}\gamma_4, \ t_{0,3}\beta_4, \ t_{0,1,3}\beta_4, \ t_{0,1,2,3}\beta_4,$$

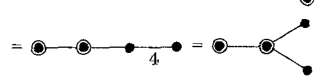
there are seven which are also derivable from [3, 4, 3] or $[3^{1,1,1}]$, namely:

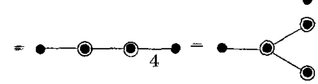
$\beta_4 = \{3, 3, 4\}$  (8 vertices),

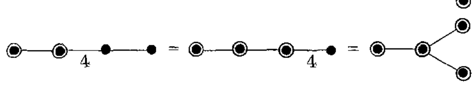
$\{3, 4, 3\} =$  $= t_1\beta_4$ (24 vertices),

$t_1\gamma_4 =$  $= t_2\beta_4$ (32 vertices),

$t_1\{3, 4, 3\} =$  $= t_{0,2}\beta_4$ (96 vertices),

$t_{0,1}\beta_4$  (48 vertices),

$t_{1,2}\beta_4$  (96 vertices),

$t_{0,1}\{3, 4, 3\} =$  $= t_{0,1,2}\beta_4$ (192 vertices).

These coincidences can easily be verified by comparing the positions of a typical vertex in the three kinds of fundamental region. In the notation of § 2.2, the appropriate positions in these seven cases are c, d, e, f, a point on dc, a point on de, and a point on df.

The six remaining polytopes derivable from the group [3, 4, 3] are

$$t_{0,2},\ t_{0,3},\ t_{1,2},\ t_{0,1,2},\ t_{0,1,3},\ t_{0,1,2,3}$$

all applied to $\{3,4,3\}$. Coordinates for their vertices can be found by taking the generators of [3, 4, 3] to be reflections in the 4 hyperplanes

$$x_1=0, \quad x_1+x_2=x_3+x_4, \quad x_2=x_3, \quad x_3=x_4.$$

The first three equations yield (0, 1, 1, 0) and hence the permutations of $(\pm 1, \pm 1, 0, 0)$, as before. But the last three yield (1, 1, 1, 1), which is a vertex of the reciprocal $\{3,4,3\}$:

$$(\pm 1, \pm 1, \pm 1, \pm 1) \text{ and } (\pm 2, 0, 0, 0) \text{ permuted.} \tag{2.51}$$

Related polytopes are easily deduced. For instance, a typical vertex of $t_{0,1,2,3}\{3,4,3\}$, being equidistant from the 4 hyperplanes, is given by

$$x_1=\frac{x_3+x_4-x_1-x_2}{2}=\frac{x_3-x_2}{\sqrt{2}}=\frac{x_4-x_3}{\sqrt{2}},$$

and the rest of the 1152 vertices can be deduced by applying the reflections (including the 'non-trivial' reflection R_2).

Clearly, [3, 4, 3] is the whole symmetry group of $\{3,4,3\}$, $t_{0,1}\{3,4,3\}$, $t_1\{3,4,3\}$, etc. But for the three which have symmetrical graphs, namely $t_{0,3}\{3,4,3\}$, $t_{1,2}\{3,4,3\}$ and $t_{0,1,2,3}\{3,4,3\}$, the symmetry group is [[3, 4, 3]], of order 2304, as it includes the half-turn T that rotates the fundamental region for [3, 4, 3] into itself. Unlike [3, 3, 3], [3, 4, 3] contains the central inversion (which reverses the sign of each coordinate) [11, p. 226], so the extended group [[3, 4, 3]] is not the direct product of [3, 4, 3] and $\mathfrak{C}_2$. For a presentation, see (2.28) with $q=4$.

The square faces of $t_{0,3}\{p,q,p\}$ form the regular skew polyhedron $\{4, 2q \mid p\}$ mentioned in connection with (2.26) [8, pp. 87–89].

For the group [5, 3, 3], of order 14400, we can use the four mirrors

$$\sqrt{5}x_1=x_2+x_3+x_4, \quad x_1=x_2, \quad x_2=x_3, \quad x_3=x_4.$$

The first three equations (with $x_1=x_2=x_3=\tau$, for convenience) yield a vertex $(\tau, \tau, \tau, \tau^{-2})$ for the 600-cell $\{3,3,5\}$. Here we are writing τ for the 'golden section' number $(\sqrt{5}+1)/2$. The complete set of 120 vertices is then found to consist of the permutations of

$(\tau, \tau, \tau, \tau^{-2})$ and $(\tau^2, \tau^{-1}, \tau^{-1}, \tau^{-1})$
with an even number of minus signs,

$(\sqrt{5}, 1, 1, 1)$ with an odd number of minus signs, and
$(\pm 2, \pm 2, 0, 0)$.

The corresponding coordinates for the reciprocal 120-cell {5, 3, 3} (using the last three equations) are so unpleasant [11, p. 240] that it seems preferable to use a different set of equations for the same four mirrors, namely

$$\tau x_1 = x_2 + \tau^{-1} x_3, \quad x_1 = 0, \quad x_1 + x_2 + x_3 + x_4 = 0, \quad x_4 = 0.$$

The last three equations now yield the familiar coordinates for {5, 3, 3} [11, p. 157], while the first three yield, for the 600-cell {3, 3, 5},

the even permutations of $(\pm\tau, \pm 1, \pm\tau^{-1}, 0)$,

the permutations of $(\pm 2, 0, 0, 0)$, and $(\pm 1, \pm 1, \pm 1, \pm 1)$

[27, pp. 211–213]. The reader may similarly find coordinates for the vertices of the rest of the fifteen polytopes in this family. These polytopes are all distinct, but three of them have each two equally simple names:

$$t_{0,3}\{5,3,3\} = t_{0,3}\{3,3,5\}, \quad t_{1,2}\{5,3,3\} = t_{1,2}\{3,3,5\},$$
$$t_{0,1,2,3}\{5,3,3\} = t_{0,1,2,3}\{3,3,5\}$$

(because the graphs can be turned upside down). Following Mrs. Boole Stott, Wythoff [31, pp. 967, 969] called these three

$$e_3\,C_{600}, \quad c e_1 e_2\, C_{600}, \quad e_1 e_2 e_3\, C_{600}.$$

Again, as for the [3, 3, 3] family, it is sometimes desirable to use *five* coordinates which can be freely permuted. Such coordinates $(u_1, u_2, u_3, u_4, u_5)$, with sum zero, can be derived from $(x_1, x_2, x_3, x_4, 0)$ by applying the transformation

$$\begin{aligned}
2u_1 &= -x_1 + \tau^2 x_2 + \tau^{-2} x_3, & \sqrt{5}x_1 &= \tau^{-1}u_2 + \tau u_3,\\
2u_2 &= \tau^{-2}x_1 - x_2 + \tau^2 x_3, & \sqrt{5}x_2 &= \tau^{-1}u_3 + \tau u_1,\\
2u_3 &= \tau^2 x_1 + \tau^{-2} x_2 - x_3, & \sqrt{5}x_3 &= \tau^{-1}u_1 + \tau u_2,\\
2u_4 &= -x_1 - x_2 - x_3 + \sqrt{5}x_4, & \sqrt{5}x_4 &= u_4 - u_5,\\
2u_5 &= -x_1 - x_2 - x_3 - \sqrt{5}x_4,
\end{aligned}$$

which is valid since it implies

$$\begin{aligned}
5x_1 &= \sqrt{5}(\tau^{-1}u_2 + \tau u_3) = (\tau^{-2}+1)u_2 + (\tau^2+1)u_3 - \Sigma u\\
&= -u_1 + \tau^{-2}u_2 + \tau^2 u_3 - u_4 - u_5, \text{ etc.,}
\end{aligned}$$

and the matrix

$$\frac{1}{\sqrt{10}}\begin{bmatrix}
-1 & \tau^{-2} & \tau^2 & -1 & -1\\
\tau^2 & -1 & \tau^{-2} & -1 & -1\\
\tau^{-2} & \tau^2 & -1 & -1 & -1\\
0 & 0 & 0 & \sqrt{5} & -\sqrt{5}\\
\sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2}
\end{bmatrix}$$

is orthogonal. In terms of the u coordinates (with sum zero), the 120 vertices of $\{3,3,5\}$ are the permutations of

$(\sqrt{5},0,0,0,-\sqrt{5})$,		20
$(\tau^2,\tau^{-2},-1,-1,-1)$,	$(1,1,1,-\tau^{-2},-\tau^2)$,	40
$(2,\tau^{-1},\tau^{-1},-\tau,-\tau)$,	$(\tau,\tau,-\tau^{-1},-\tau^{-1},-2)$,	60
		120

while the 600 vertices of $\{5,3,3\}$ are the permutations of

$(4,-1,-1,-1,-1)$,	$(1,1,1,1,-4)$,	10
$(\sqrt{5},\sqrt{5},0,-\sqrt{5},-\sqrt{5})$,		30
$(\tau^{-1},\tau^{-1},\tau^{-1},2,-\sigma)$,	$(\sigma,-2,-\tau^{-1},-\tau^{-1},-\tau^{-1})$,	40
$(\tau,\tau,\tau,-2,-\sigma')$,	$(\sigma',2,-\tau,-\tau,-\tau)$,	40
$(\tau\sqrt{5},0,0,-\tau^{-1}\sqrt{5},-\sqrt{5})$,	$(\sqrt{5},\tau^{-1}\sqrt{5},0,0,-\tau\sqrt{5})$,	120
$(2,2,\tau^{-1},-\tau,-3)$,	$(3,\tau,-\tau^{-1},-2,-2)$,	120
$(\tau^2,2\tau^{-1},\tau^{-2},-1,-2\tau)$,	$(2\tau,1,-\tau^{-2},-2\tau^{-1},-\tau^2)$,	240
		600

where $\sigma=(3\sqrt{5}+1)/2$ and $\sigma'=(3\sqrt{5}-1)/2=\sigma-1$ [11, pp. 240, 301].

It is interesting to observe that the first row of this last table exhibits two of the 120 regular simplexes $\alpha_4=\{3,3,3\}$ that can be inscribed in $\{5,3,3\}$ [11, p. 304 ('Section 18_0')]. The second row exhibits an inscribed $t_{1,2}\alpha_4$ similar to the one given by the permutations of $(1,1,0,-1,-1)$ or $(2,2,1,0,0)$ [14, p. 127]. Similarly, the first 20 vertices of the above $\{3,3,5\}$ belong to a $t_{0,3}\alpha_4$ similar to the one given by the permutations of $(1,0,0,0,-1)$ [6, p. 470–475].

Since the 10 pairs of opposite vertices of $t_{0,3}\alpha_4$ are interchanged by the 10 reflections that occur in [3, 3, 3], while the 60 pairs of opposite vertices of $\{3,3,5\}$ are interchanged by the 60 reflections that occur in [5, 3, 3], the subgroup $[3,3,3]\cong\mathfrak{S}_5$ of [5, 3, 3] is thus seen to be generated by a subset of those 60 reflections.

2.6. Four-Dimensional Honeycombs

We return now to the infinite reflection groups listed at the end of § 2.2. For the group $\tilde{A}_4$, symbolized by a pentagon, the 5 mirrors may be taken to be

$$u_1=u_2,\quad u_2=u_3,\quad u_3=u_4,\quad u_4=u_5,\quad u_5=u_1-1 \tag{2.61}$$

in the 4-space $\Sigma u_\nu=0$. More precisely, the fundamental region is the Euclidean 4-simplex

$$u_1\geqq u_2\geqq u_3\geqq u_4\geqq u_5\geqq u_1-1$$

[18, pp. 151–153]. The first four reflections generate the symmetric subgroup $[3,3,3]\cong\mathfrak{S}_5$ that permutes the five coordinates, while the fifth takes

$(u_1, u_2, u_3, u_4, u_5)$ to $(u_5+1, u_2, u_3, u_4, u_1-1)$. Thus the whole group $\tilde{A}_4$ allows us to add 1 to one coordinate while subtracting 1 from another. Solving the first four of the Eq. (2.61) along with $\Sigma u_\nu = 0$, we see that the fifth vertex of the simplex is the origin $(0,0,0,0,0)$ whose orbit consists of all the points whose 5 coordinates are *integers with sum zero*. These are the vertices of the 'alphahedroid' $\alpha_4 h$ whose symbol is a pentagon with a ring round one vertex [4, p. 135; 5, p. 366]. Since the difference of any two such points, regarded as vectors, belongs to the set, the vertices of $\alpha_4 h$, like those of $\alpha_2 h = \{3,6\}$ and $\alpha_3 h = \mathrm{h}\,\delta_4$ (see § 1.9, p. 402), form a *lattice*.

Six other honeycombs can be obtained by putting rings round two vertices of the pentagon, adjacent or non-adjacent, or round three vertices, consecutive or separated, or round four vertices, or all five [8, p. 73]. In each case a typical vertex arises as the solution of a set of four equations. For instance, when all the five vertices are ringed we have

$$u_1 - u_2 = u_2 - u_3 = u_3 - u_4 = u_4 - u_5 = u_5 - u_1 + 1,$$

and the typical vertex is $(\frac{2}{5}, \frac{1}{5}, 0, -\frac{1}{5}, -\frac{2}{5})$ or, by a convenient change of scale (so that now we can add 5 to one coordinate while subtracting 5 from another),

$$(2, 1, 0, -1, -2).$$

This is Hinton's honeycomb [21, p. 225] whose vertices are given by all sets of 5 integers mutually incongruent modulo 5, with sum zero. Thus every cell is a 'permutohedron' $\mathrm{t}_{0,1,2,3}\,\alpha_4$.

When the number of rings is less than 5, the various kinds of cell are other derivatives of the regular simplex α_4; their symbols are derived by omitting each vertex of the simplex in turn [14, pp. 128–129].

For the lattice $\alpha_{n-1} h$, symbolized by an n-gon with one ring, the simpler name $0_{[n]}$ is suggested by analogy with $[3^{[n]}]$ and 0_{ij} [14, p. 131]. Thus $0_{[3]} = \{3,6\}$ and $0_{[4]} = \mathrm{h}\,\delta_4$.

Since the 4-dimensional cubic lattice $\delta_5 = \{4,3,3,4\}$ is formed by the points whose coordinates are any 4 integers, a characteristic orthoscheme has for its 5 vertices the origin $(0,0,0,0)$, the midpoint $(\frac{1}{2},0,0,0)$ of the edge going to $(1,0,0,0)$, the centre $(\frac{1}{2},\frac{1}{2},0,0)$ of a square $\{4\}$, the centre $(\frac{1}{2},\frac{1}{2},\frac{1}{2},0)$ of a cube $\{4,3\}$, and the centre $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ of a cell $\{4,3,3\}$. Thus the five mirrors for the reflection group $\tilde{C}_4 = [4,3,3,4]$ are

$$x_1 = \tfrac{1}{2}, \quad x_1 = x_2, \quad x_2 = x_3, \quad x_3 = x_4, \quad x_4 = 0; \tag{2.62}$$

more precisely, the fundamental region is given by

$$\tfrac{1}{2} \geqq x_1 \geqq x_2 \geqq x_3 \geqq x_4 \geqq 0.$$

The 'middle' reflections R_2, R_3, R_4 generate the symmetric subgroup $[3,3] \cong \mathfrak{S}_4$ that permutes the four coordinates, while R_5 reverses the sign of x_4; thus any sign can be reversed. The product of the reversal of x_1 with R_1 is the translation that adds 1 to x_1.

As an instance of Wythoff's construction, let us find a typical vertex of $t_{0,1}\,\delta_5$ by solving the four equations

$$\tfrac{1}{2}-x_1=(x_1-x_2)/\sqrt{2}, \qquad x_2=x_3=x_4=0.$$

The result, $(1-\sqrt{\tfrac{1}{2}},0,0,0)$, shows that the whole set of vertices is given by 4 integers to just one of which $\pm\sqrt{\tfrac{1}{2}}$ is added.

Among the vertices of δ_5, we can pick out half by insisting that the integral coordinates have an *even sum*. We thus obtain the 'alternated' lattice

$$h\,\delta_5=\{3,3,4,3\} \tag{2.63}$$

of edge-length $\sqrt{2}$ [11, p. 158]. The edges through $(0,0,0,0)$ go to the 24 points $(\pm 1, \pm 1, 0, 0)$ permuted, which are the vertices of $\{3,4,3\}$, in agreement with the Schläfli symbol $\{3,3,4,3\}$.

Applying the similarity $(x_1,x_2,x_3,x_4)\to(x_1+x_2, x_1-x_2, x_3+x_4,\ x_3-x_4)$, we obtain a larger $\{3,3,4,3\}$, of edge-length 2, whose vertices have coordinates which all even or all odd, those joined to the origin being (2.51). A typical cell $\beta_4=\{3,3,4\}$ has the 8 vertices

$$(0,0,0,0), \quad (2,0,0,0), \quad (2,2,0,0), \quad (0,2,0,0,), \quad (1,1,\pm 1,\pm 1).$$

Since the centre of this β_4 is $(1,1,0,0)$, the vertices of the reciprocal honeycomb $\{3,4,3,3\}$ are all the points whose coordinates are *two odd and two even*. These points do not form a lattice, but they can be derived from the 'small' lattice $\{3,3,4,3\}$ (coordinates with an even sum) by deleting the vertices of a larger $\{3,3,4,3\}$ (coordinates all odd or all even) which is reciprocal to the $\{3,4,3,3\}$. In other words, the set of vertices of $\{3,4,3,3\}$ is the 'difference' between two similar, but not homothetic, lattices $\{3,3,4,3\}$, just as, in two dimensions, the set of vertices of the non-lattice tessellation $\{6,3\}$ is the 'difference' between two similar, but not homothetic, lattices $\{3,6\}$ [11, p. 64].

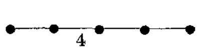
Since $\{3,4,3,3\}$ has for its vertices the permutations of $(1,1,0,0)$ (mod 2), its characteristic orthoscheme has one vertex $(1,1,0,0)$, another at the midpoint $(1,\tfrac{1}{2},\tfrac{1}{2},0)$ of the edge going to $(1,0,1,0)$, a third at the centre $(1,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3})$ of the triangle $\{3\}$ joining this edge to $(1,0,0,1)$, a fourth at the centre $(1,0,0,0)$ of an octahedron $\{3,4\}$ in the hyperplane $x_1=1$, and a fifth at the centre $(0,0,0,0)$ of the cell $\{3,4,3\}$ given by the permutations of $(\pm 1, \pm 1, 0, 0)$. The facets of this orthoscheme

$$(1,1,0,0)\,(1,\tfrac{1}{2},\tfrac{1}{2},0)\,(1,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3})\,(1,0,0,0)\,(0,0,0,0)$$

lie in the hyperplanes

$$x_2=x_3, \quad x_3=x_4, \quad x_4=0, \quad x_1=x_2+x_3+x_4, \quad x_1=1. \tag{2.64}$$

These are the mirrors for the reflections R_1, R_2, R_3, R_4, R_5, represented by the dots in the graph



for the group $\tilde{F}_4 \cong [3,4,3,3]$. Since R_3 transforms $x_3 = x_4$ into $x_3 + x_4 = 0$, and R_4 transforms this into $x_1 = x_2$, the group includes the transposition

$$(1\ 2) = R_4 R_3 R_2 R_3 R_4$$

as well as $(2\ 3) = R_1$ and $(3\ 4) = R_2$. Thus all the coordinates can be permuted. The product of the reflections in $x_1 = 0$ and $x_1 = 1$ (the latter being R_5) is the translation that adds 2 to x_1. Thus we can add any even number to any coordinate.

Related honeycombs are easily obtained by solving suitable linear equations. For instance, a typical vertex of $t_{0,1}\{3,4,3,3\}$, given by

$$x_2 - x_3 = x_3 - x_4, \qquad x_4 = 0, \qquad x_2 + x_3 + x_4 = x_1 = 1,$$

is $(1, \frac{2}{3}, \frac{1}{3}, 0)$. Multiplying by 3 to avoid fractions, and allowing reversals of sign, we see that the vertices of $t_{0,1}\{3,4,3,3\}$ are all the permutations of

$$(3, \pm 2, \pm 1, 0) \quad (\text{mod } 6). \tag{2.65}$$

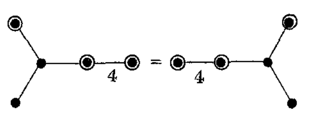
From the group $\tilde{B}_4 = \left\{4, 3, \begin{matrix}3\\3\end{matrix}\right\}$, with mirrors

$$x_1 = \tfrac{1}{2}, \qquad x_1 = x_2, \qquad x_2 = x_3, \qquad \begin{matrix} x_3 = x_4, \\ x_3 + x_4 = 0, \end{matrix} \tag{2.66}$$

we obtain an alternative symbol for $t_1 \delta_5$ by putting rings on both the 'special' dots. If instead we ring just one of them, we obtain the alternated lattice $h\delta_5$ (2.63). By analogy with the notation in the table on p. 403 of Part I, we can insert extra rings to obtain

$$h_2\delta_5, \quad h_3\delta_5, \quad h_4\delta_5, \quad h_{2,3}\delta_5, \quad h_{2,4}\delta_5, \quad h_{3,4}\delta_5, \quad h_{2,3,4}\delta_5.$$

For instance, $h_{3,4}\delta_5$ is



Because the four groups $\tilde{D}_4$, $\tilde{B}_4$, $\tilde{C}_4$, $\tilde{F}_4$ are intimately related (see the end of § 2.2), the corresponding honeycombs can often be derived several ways. Along with (2.63), we have

$$\begin{aligned} h_2\delta_5 &= t_{0,1}\{3,3,4,3\}, & t_2\delta_5 &= \{3,4,3,3\} = t_1\{3,3,4,3\}, \\ h_{2,3}\delta_5 &= t_{1,2}\{3,3,4,3\}, & t_{1,3}\delta_5 &= t_1\{3,4,3,3\}, \\ h_3\delta_5 &= t_2\{3,3,4,3\}, & t_{1,2,3}\delta_5 &= t_{0,1}\{3,4,3,3\}. \end{aligned}$$

This list includes all the honeycombs that could be derived from the group $\tilde{D}_4$, whose mirrors may be taken to be

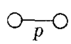
$$\begin{array}{ccc} x_1=0, & & x_4=0, \\ & \Sigma x_\nu=1, & \\ x_2=0, & & x_3=0 \end{array}$$

[8, p. 48].

2.7. The Four-Dimensional Analogues of the Snub Cube

Instead of using the whole reflection group G, as in §2.4, we can sometimes derive a uniform polytope as the convex hull of the transforms of a point Q under the rotation subgroup G^+. If Q lies on one or more of the mirrors, this orbit of Q is the same as if we had used the whole group G; so now we may assume that Q does *not* lie on any mirror but is an interior point of the spherical simplex which is a fundamental region for G. The convex hull of the orbit (for G^+) of such a point Q may be described as a *general snub polytope*. Among all the finite reflection groups G, we seek those which admit a position of Q yielding a *uniform* snub polytope.

From the graph that symbolizes G, we derive an appropriate symbol for the general snub polytope by changing every dot into a ring, as in §1.5 (p. 394). For instance, when the mirrors are two intersecting lines in a plane, or two points on a circle, the snub polytope is a p-gon



which is (in this exceptional case) regular for *all* positions of Q.

As we saw in §1.2 and §2.2, the n mirrors and their transforms by G decompose the $(n-1)$-sphere into congruent $(n-1)$-dimensional spherical simplexes, any one of which will serve as a fundamental region for G. We may describe these simplexes as being alternately white and black (or 'shaded', as on p. 392 of §1.5), so that two of the same colour are *directly* congruent while two of opposite colours (such as two that are adjacent) are *oppositely* congruent. If Q lies inside a white simplex, the remaining vertices of the general snub polytope Π_n are corresponding points in all the other white simplexes. The number of vertices of Π_n, being equal to the number of white simplexes, is half the order of G; that is, the number of vertices is equal to the order of G^+.

Every black simplex is surrounded by n white simplexes, each containing a vertex of Π_n. These n vertices form an $(n-1)$-simplex Σ_{n-1} which is one of the facets of Π_n. The number of facets of this type, being equal to the number of black simplexes, is again half the order of G. Another type of facet (occurring whenever $n>2$) is an $(n-1)$-dimensional snub polytope Π_{n-1}, derived from $n-1$ of the n mirrors, namely those which pass through any particular vertex P of the white simplex containing Q. The symbol for Π_{n-1} is derived from the symbol for Π_n by removing one of the n rings (along with any links which emanate from that ring). The number of facets of this type (for each particular ring) is equal to the index of the appropriate subgroup of G.

Every vertex of Π_n, such as Q, belongs to n facets Π_{n-1} (of various kinds) corresponding to the n vertices of the fundamental region or to the n rings in the symbol. Q also belongs to n facets Σ_{n-1}, corresponding to the adjacent black simplexes.

The case when $n=2$ is exceptional, because the one-dimensional snub polytope, represented by a single ring, collapses to a point. The Σ_1's and Π_1's of the p-gon Π_2 are its sides and vertices.

When G is [4,3] (or [3,4]), of order 48, we have the snub cube

$$\circ\underset{4}{—}\circ—\circ$$

or $s\begin{Bmatrix}4\\3\end{Bmatrix}$, which might more consistently have been named 'snub cuboctahedron', and we verify that it has 24 vertices, 24 triangles Σ_2, $48/8=6$ squares $\circ\underset{4}{—}\circ$, and $48/6=8$ triangles $\circ—\circ$ (making 24+8) triangles altogether). The remaining faces of type Π_2 (symbolized by the two outermost rings alone) are digons which we naturally regard as collapsing to form $48/4=12$ edges, each separating two triangles Σ_2. The remaining 48 edges consist of the 24 sides of the 6 squares and the 24 sides of the 8 triangles Π_2.

The 24 vertices of $s\begin{Bmatrix}4\\3\end{Bmatrix}$ are especially significant as being the solution, for $m=24$, of the Problem of Tammes [29] which asks for the distribution of m points on a sphere so as to maximize the minimal distance between pairs of them. It has been solved for $m \leqq 12$ but not for $12 < m < 24$.

Since snub polygons are necessarily regular, a snub polyhedron is uniform whenever its triangles Σ_2 are equilateral. It follows (by induction) that a snub polytope Π_n is uniform whenever its facets Σ_{n-1} are *regular* simplexes, for then its facets Π_{n-1} are automatically uniform and all its 2-faces are regular.

To ensure uniformity of the general snub polytope Π_n, the location of the initial point Q is, in the 2-dimensional case, arbitrary; in the 3-dimensional case, it is just determinate, as we saw in §1.5 (on p. 393). In more than 3 dimensions, it is nowhere to be found, except in a few special cases; for, as we shall soon see, the conditions to be satisfied impose certain restrictions on the angles $\pi/q_{\mu\nu}$ of the fundamental region (see (2.11)). The crucial case is in 4 dimensions, since the graph for a 5-dimensional group is admissible only if every subgraph derived from it by removing a single dot symbolizes one of the admissible groups in 4 dimensions. This condition (for the uniformity of a 5-dimensional snub polytope) will be seen to be sufficient as well as necessary.

In any snub polytope Π_n, let $Q_1 Q_2 \ldots Q_n$ be a facet Σ_{n-1}. The vertices Q_ν, being similarly situated points within all the white simplexes that surrounded one black simplex, are derivable from a single point Q' within the black simplex by reflecting in the n hyperplanes which determine that simplex. (Of course Q', lying within the *black* simplex, is *not* a vertex of Π_n.) If Π_n is to be uniform, the simplex $Q_1 Q_2 \ldots Q_n$ must be regular, and so too must the similar simplex $M_1 M_2 \ldots M_n$ (of half the linear size) whose vertices are the midpoints of the straight line-segments $Q'Q_\nu$. In other words, the snub polytope Π_n is uniform only if there exists a point Q' whose orthogonal projections M_ν on the n hyperplanes are the vertices of a *regular* simplex.

Let $t_1, t_2, \ldots, t_n$ denote the (Euclidean) distances of Q' from the n hyperplanes. In the plane triangle $Q' M_\mu M_\nu$, the sides $t_\mu = Q' M_\mu$ and $t_\nu = Q' M_\nu$ form an angle $\pi - \pi/q_{\mu\nu}$ whose cosine is $-c_{\mu\nu}$ (see (1.35)). Hence

$$M_\mu M_\nu^2 = t_\mu^2 + t_\nu^2 + 2 c_{\mu\nu} t_\mu t_\nu \qquad (\mu \neq \nu). \tag{2.71}$$

But $M_\mu M_\nu$ is a typical edge of the simplex $M_1 M_2 \ldots M_n$ whose regularity is desired. Thus the n unknown (positive) lengths $t_1, t_2, \ldots, t_n$ should satisfy the $\binom{n}{2} - 1$ equations that arise by equating the $\binom{n}{2}$ expressions (2.71) (with $\mu < \nu$, say) to one another. Since the equations are homogeneous, and we are concerned only with the mutual ratios of the n x's, the problem has (as we already know from § 1.5) a unique solution when $n = 3$. But when $n > 3$, so that $\binom{n}{2} > n$, the equations are consistent only if the coefficients $c_{\mu\nu}$ take certain special values.

A closely analogous theory of snub *honeycombs* arises when we consider infinite discrete groups generated by n reflections in Euclidean $(n-1)$-space. The condition for uniformity is still the consistency of such equations, only now the x's, instead of being oblique coordinates based on n concurrent hyperplanes, are distances from the facets of a Euclidean $(n-1)$-simplex; for instance, they are trilinear coordinates when $n = 3$, 'tetrahedral' coordinates when $n = 4$.

The theory can be modified to cover reducible groups whose fundamental regions are Cartesian products of two or more simplexes. For instance, in the case of the first half of (1.91) (on p. 401), Σ_3 is not a single regular tetrahedron but a pair of such tetrahedra forming together a triangular bipyramid: the reciprocal of the fundamental region (which is a triangular prism). This snub honeycomb is $\mathrm{h}(\{6,3\} \times \{\infty\})$.

Let us now examine the various groups G with $n = 4$, both finite in 4 dimensions and infinite in 3, so as to determine which of the corresponding snub figures are uniform.

Consider first the groups $[p,q,r]$, where we allow p,q,r to take the value 2 so as to admit the reducible groups whose graphs are disconnected. Since now $c_{13} = c_{14} = c_{24} = 0$, the equations reduce to

$$\begin{aligned} t_1^2 + t_2^2 + 2c_{12} t_1 t_2 = t_2^2 + t_3^2 + 2c_{23} t_2 t_3 = t_3^2 + t_4^2 + 2c_{34} t_3 t_4 \\ = t_1^2 + t_3^2 = t_1^2 + t_4^2 = t_2^2 + t_4^2, \end{aligned}$$

whence $t_3 = t_4$, $t_1 = t_2$, $c_{23} = 0$ and

$$2(1 + c_{12}) t_1^2 = t_1^2 + t_3^2 = 2(1 + c_{34}) t_3^2.$$

Elimination of t_1/t_3 yields $c_{12} + c_{34} + 2c_{12} c_{34} = 0$. Since the angle $\pi/q_{\mu\nu}$, whose cosine is $c_{\mu\nu}$, cannot be obtuse, it follows that $c_{12} = c_{34} = 0$. Thus the only admissible group $[p,q,r]$ is $[2,2,2]$, of order 16, whose graph consists of 4 separate dots. The polytope

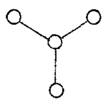
$$\circ \quad \circ \quad \circ \quad \circ \tag{2.72}$$

has 8 vertices, and its facets are 8 tetrahedra Σ_3 and 8 tetrahedra Π_3 (symbolized by 3 separate rings). It is obviously the regular 16-cell $\beta_4=\{3,3,4\}$. Analogously, the n-dimensional polytope symbolized by n separate rings is the half-measure polytope $\mathrm{h}\,\gamma_n$ [11, p. 156]. Its 2^{n-1} vertices are alternate vertices of the measure polytope $\gamma_n=\{4,3,\ldots,3\}$ which is symbolized by n separate ringed dots.

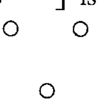
The finite group $[3^{1,1,1}]=\begin{bmatrix}3\\3\\3\end{bmatrix}$ and the infinite group $[4,3^{1,1}]$ (see §1.3, p. 386) may be considered together by writing $c_{14}=c_{24}=\frac{1}{2}$, $c_{23}=c_{13}=c_{12}=0$. The equations

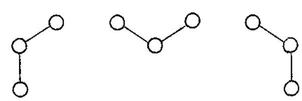
$$t_1^2+t_4^2+t_1t_4=t_2^2+t_4^2+t_2t_4=t_3^2+t_4^2+2c_{34}t_3t_4$$
$$=t_2^2+t_3^2=t_1^2+t_3^2=t_1^2+t_2^2$$

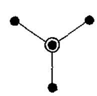
imply $t_1=t_2=t_3$ and $c_{34}=\frac{1}{2}$. Thus the snub *polytope*


(2.73)

or $\mathrm{s}\begin{Bmatrix}3\\3\\3\end{Bmatrix}$, or (more conveniently) $\mathrm{s}\{3,4,3\}$, *is* uniform (with $t_1=t_2=t_3=\tau t_4$) but the snub *honeycomb* (with one link marked 4) is not. Some 'nearly uniform' varieties of the latter were described by Merkel [24; see also 13]. Similarly, the remaining infinite group $[3^{[4]}]$ yields another set of inconsistent equations and a 'nearly uniform' honeycomb of irregular tetrahedra and irregular icosahedra. We conclude that a uniform snub honeycomb in 3 dimensions cannot arise from a tetrahedral fundamental region but only from a triangular prism or a cube, as in (1.91) (see p. 401, 402). In both cases the cells surrounding a vertex consist of 8 tetrahedra and 6 octahedra.

Since the order of $[3^{1,1,1}]$ is 192, $\mathrm{s}\{3,4,3\}$ has 96 vertices, 96 tetrahedra Σ_3, $192/8=24$ tetrahedra  and 3 sets of $192/24=8$ icosahedra



(making 24 icosahedra altogether). Comparing (2.73) with the symbol  for $\{3,4,3\}$, and considering two specimens of the pyramidal fundamental region, one white and one black, with a common base, we see that each vertex of $\mathrm{s}\{3,4,3\}$ lies on an edge of $\{3,4,3\}$, but not at the midpoint of the edge. In fact, the 96 vertices of $\mathrm{s}\{3,4,3\}$ divide the 96 edges of the 24-cell $\{3,4,3\}$

according to the golden ratio $\tau:1$ or $\tau^{-1}:\tau^{-2}$ $(\tau^{-1}+\tau^{-2}=1)$ [11, p. 152]. In the 24-cell given by permutations of $(\pm 1, \pm 1, 0, 0)$, the typical edge $(1,1,0,0)$ $(1,0,1,0)$ is so divided at the point

$$(1, \tau^{-1}, \tau^{-2}, 0).$$

Thus the 96 vertices of the *snub* 24-*cell* $s\{3,4,3\}$ are given by the *even* permutations of $(\pm 1, \pm\tau^{-1}, \pm\tau^{-2}, 0)$ or, after a suitable dilatation,

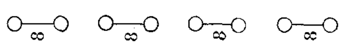
$$(\pm\tau, \pm 1, \pm\tau^{-1}, 0) \tag{2.74}$$

[11, p. 157].

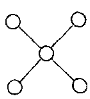
Since the base of the pyramidal fundamental region lies in the hyperplane that perpendicularly bisects an edge of $\{3,4,3\}$, and R_1 is the reflection in that hyperplane, the symmetry group of $s\{3,4,3\}$ lacks R_1 and R_2 and is precisely $[3^+,4,3]$, of order 576.

Since each vertex of $s\{3,4,3\}$ belongs to 5 tetrahedra and 3 icosahedra, the vertex figure [11, p. 163] is an irregular polyhedron whose faces consist of 5 triangles and 3 pentagons. The pentagons surrounded one of the triangles, and there is an 'opposite' triangle surrounded by the remaining 3 triangles. Thus the vertex figure of $s\{3,4,3\}$ can be derived from the regular icosahedron $\{3,5\}$, which is the vertex figure of the 600-cell $\{3,3,5\}$, by cutting off pentagonal pyramids from three non-adjacent corners. Analogously, the whole $s\{3,4,3\}$ can be derived from $\{3,3,5\}$ by cutting off icosahedral 'pyramids' from 24 of the 120 'corners', namely the vertices of an inscribed $\{3,4,3\}$ [11, p. 272].

We pass on to the case when $n=5$. Here there is no need to use coordinates; for, a snub polytope in 5 dimensions, or a snub honeycomb in 4, cannot be uniform unless every facet of type Π_4 is uniform, that is, either a β_4 (2.72) or an $s\{3,4,3\}$ (2.73). Thus the only uniform snub polytope is the $h\gamma_5$ symbolized by 5 separate rings, and the only uniform snub honeycombs are



and



(2.75)

For the former, G is $\mathfrak{D}_\infty \times \mathfrak{D}_\infty \times \mathfrak{D}_\infty \times \mathfrak{D}_\infty$ and its fundamental region is the 4-cube $\gamma_4 = \gamma_1^4$ [11, p. 124]. The snub honeycomb is $h\delta_5 = \{3,3,4,3\}$, having one vertex at the centre of each white cell γ_4 in a 4-dimensional chess board [11, pp. 156, 158]. It has, at each vertex, 24 cells $h\gamma_4 = \beta_4 = \{3,3,4\}$. Of these 24 β_4's, 8 are reciprocal to the 8 black γ_4's that surround the initial white γ_4. Each of the remaining 16 is symbolized by 4 separate rings, one from each piece of the disconnected symbol for the whole snub honeycomb. There is an analogous $(n-1)$-dimensional snub honeycomb $h\delta_n$ for any value of n, but when $n>5$ it is no longer regular: its cells consist of β_{n-1}'s, $2(n-1)$ at each vertex, and $h\gamma_{n-1}$'s, 2^{n-1} at each vertex.

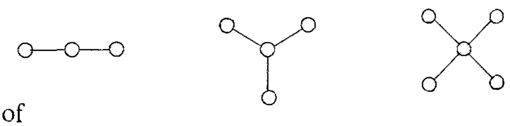
Finally, for the more interesting snub honeycomb whose symbol consists of one ring linked to four others, G is $[3^{1,1,1,1}] = \tilde{D}_4$ and its (Euclidean) fundamental region is a 4-dimensional tetrahedral pyramid such as might be cut off symmetrically from one corner of a 4-cube γ_4. When two such pyramids, one white and one black, are placed base-to-base, their two apices form an edge of the regular honeycomb $\{3,4,3,3\}$; thus each vertex of

$$\mathrm{s}\left\{\begin{matrix}3\\3\\3\\3\end{matrix}\right\}$$

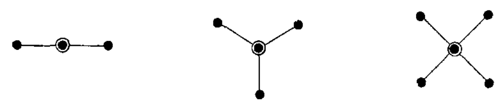
lies on an edge of $\{3,4,3,3\}$, but not at the midpoint of the edge, and the awkardly tall symbol may reasonably be reduced to

$$\mathrm{s}\{3,4,3,3\}.$$

The $\mathrm{s}\{3,4,3\}$'s of $\mathrm{s}\{3,4,3,3\}$ are inscribed in the $\{3,4,3\}$'s of $\{3,4,3,3\}$, just as the icosahedra of $\mathrm{s}\{3,4,3\}$ are inscribed in the octahedra of $\{3,4,3\}$. In fact, the vertices of



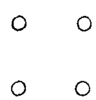
divide the edges of



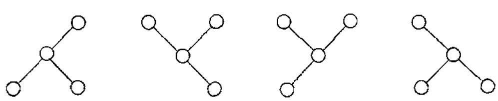
according to the golden ratio. In the $\{3,4,3,3\}$ given by permutations of $(1,1,0,0)$ (mod 2) [11, p. 158], the typical edge $(1,1,0,0)\,(1,0,1,0)$ is so divided at the point $(1,\tau^{-1},\tau^{-2},0)$. Thus the vertices of the snub honeycomb $\mathrm{s}\{3,4,3,3\}$ are given by the even permutations of $(1,\pm\tau^{-1},\pm\tau^{-2},0)$ (mod 2) or, since $\tau^{-2}=2-\tau$,

$$(\pm\tau,1,\pm\tau^{-1},0) \quad (\mathrm{mod}\,2). \tag{2.76}$$

Each vertex of this honeycomb is surrounded by 5 simplexes $\Sigma_4=\alpha_4$, one β_4



and 4 snub 24-cells



Each α_4 shares its 5 tetrahedral cells with one β_4 and 4 $\mathrm{s}\{3,4,3\}$'s. Each β_4 shares its 16 tetrahedral cells with 8 α_4's and 8 $\mathrm{s}\{3,4,3\}$'s, arranged alternately. Each $\mathrm{s}\{3,4,3\}$ shares its 24 icosahedral cells with other $\mathrm{s}\{3,4,3\}$'s, its 96 tetrahedra Σ_3 with α_4's, and its 24 tetrahedra Π_3 with β_4's.

The circuits of solids and cells surrounding the three types of triangular face are as follows:

$$\begin{array}{cccccc} \{3,5\}, & s\{3,4,3\}, & \{3,5\}, & s\{3,4,3\}, & \{3,5\}, & s\{3,4,3\}; \\ \{3,5\}, & s\{3,4,3\}, & \alpha_3, & \alpha_4, & \alpha_3, & s\{3,4,3\}; \\ \alpha_3, & \alpha_4, & \alpha_3, & \beta_4, & \alpha_3, & s\{3,4,3\}. \end{array}$$

The last case is particularly interesting because the angle between two adjacent α_3's of $s\{3,4,3\}$ is precisely the dihedral angle of $\{3,3,5\}$; we thus have a geometric 'explanation' for the curious fact that the dihedral angles of the three regular polytopes

$$\{3,3,3\}, \quad \{3,3,4\}, \quad \{3,3,5\}$$

[11, p. 293] add up to 2π.

We conclude that the only uniform snub polytopes are

$$s\{p\}=\{p\}, \quad s\begin{Bmatrix} p \\ 2 \end{Bmatrix}, \quad s\begin{Bmatrix} 3 \\ 3 \end{Bmatrix}=\{3,5\}, \quad s\begin{Bmatrix} 4 \\ 3 \end{Bmatrix}, \quad s\begin{Bmatrix} 5 \\ 3 \end{Bmatrix},$$

$$h\gamma_n \ (=\alpha_3 \text{ when } n=3,\ \beta_4 \text{ when } n=4), \qquad s\begin{Bmatrix} 3 \\ 3 \\ 3 \end{Bmatrix}=s\{3,4,3\}$$

(with p, $2p$, 12, 24, 60, 2^{n-1}, 96 vertices), and the only uniform snub honeycombs are

$$h\delta_{n+1} \ (=\delta_3 \text{ when } n=2),$$

$$s\begin{Bmatrix} 4 \\ 4 \end{Bmatrix}, \quad s\begin{Bmatrix} 6 \\ 3 \end{Bmatrix}, \quad h\{6,3\}=\{3,6\}, \quad h(\{6,3\}\times\{\infty\}), \quad s\{3,4,3,3\}.$$

Although the general construction uses a rotation group, only $s\begin{Bmatrix} p \\ 3 \end{Bmatrix}$ ($p=4,5,6$) are chiral; all the rest are reflexible. For instance, among the reflections $R_1, \ldots, R_5$ that generate $[3,4,3,3]$, the first two are missing from the symmetry group of $s\{3,4,3,3\}$. For this infinite group, generated by $R_1 R_2$, R_3, R_4, R_5, the appropriate symbol is, of course, $[3^+,4,3,3]$.

2.8. The Grand Antiprism

In §2.5 we saw how to construct many uniform polytopes by applying Wythoff's construction to reflection groups. In §2.7 we found a few more by using rotation subgroups of some of the reflection groups. Is the list of uniform polytopes in 4 dimensions now complete? To answer this question, J.H. Conway considered all the possible ways to arrange uniform polyhedra round one common vertex so that, round each edge, the sum of the dihedral angles is less than 2π. His conclusion was that the answer is No: the list is nearly complete but not quite! There is just one more polytope. In his enthusiasm he named it *the grand antiprism.* It has 100 vertices, 500 edges, 20 pentagons, 700 triangles, 20 pentagonal antiprisms and 300 tetrahedra. The vertices, edges, triangles and tetrahedra occur among the 120 vertices, 720 edges, 1200 faces and 600 cells of the regular polytope $\{3,3,5\}$. Each vertex is surrounded by 2 pentagonal antiprisms (with a common pentagonal face) and 12 tetrahedra (arranged like 12 of the 20 triangular faces of the icosahedron which is the vertex figure $\{3,5\}$ of the 600-cell $\{3,3,5\}$). In other words, the vertex figure of the grand antiprism is an irregular 14-hedron derived from the icosahedron by removing the 8 triangles that surround two adjacent vertices and inserting instead two trapezia with sides $1,1,1,\tau$.

We recall [11, pp. 247, 275, 277] that the 600-cell has pairs of equatorial decagons, such as

$$A_0 A_6 A_{12} A_{18} \ldots A_{54} \quad \text{and} \quad D_0 D_6 D_{12} D_{18} \ldots D_{54},$$

lying in completely orthogonal planes. Each decagon is surrounded by 150 tetrahedra: 5 round each of the 10 edges and 10 round each of the 10 vertices. To derive the grand antiprism, we remove the 300 tetrahedra that surround one such pair of decagons and replace them by a pair of 'necklaces', N and N', each composed of 10 pentagonal antiprisms placed base to base.

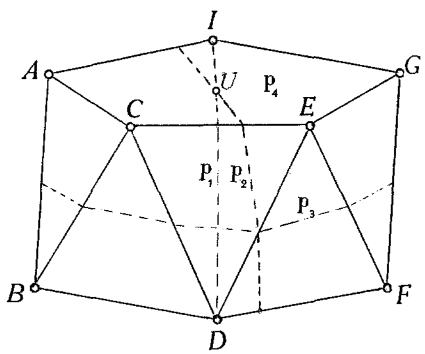


Fig. 2.8A. One of the Antiprisms

Let $ABC \ldots J$ as in Fig. 2.8A, be one of the antiprisms in the necklace N. Let U be the centre of the 'top' pentagon $ACEGI$, and V the centre of the whole antiprism. Let p_1 and p_2 be two planes through the pentagonal axis UV and through the midpoints of CE and DE respectively. Let p_3 be the plane midway between the two pentagons (that is, through the midpoints of AB, BC, CD, $DE, \ldots$), and p_4 the plane of the pentagon $ACEGI$. Thus the 4 planes p_ν form a kind of wedge.

If we regard this construction as taking place in spherical 3-space (that is, on the circumsphere of the polytope), the axis UV is a great circle. The planes p_1 and p_2, through UV, still form a dihedral angle $\pi/10$, but the planes p_3 and p_4, perpendicular to UV, instead of being parallel, intersect at an angle $\pi/10$ along the polar great circle (lying in the completely orthogonal plane through the centre of the 3-sphere). The planes (that is, great spheres) p_1 and p_2, are orthogonal to this polar great circle and cut it in points U' and V' which play the roles of U and V in one of the pentagonal antiprisms of the other necklace N'. Thus the 4 planes p_ν are the faces UVU', UVV', $VU'V'$, $UU'V'$ of a spherical tetrahedron

— the fundamental region for [10, 2, 10]. As its faces are congruent isosceles triangles, $UVV'U'$ is the kind of tetrahedron that is called a *tetragonal disphenoid* ('double wedge'). In terms of the reflections R_ν in the planes p_ν, the polytope is symmetrical by the reflection R_1, the half-turn $R_2 R_3$ (which reverses the edge DE) and the reflection R_4. These generate the group $[10, 2^+, 10]$ of order 200. But there is also the half-turn about the join of the midpoints of UU' and VV', which interchanges the two necklaces. Thus the complete symmetry group of the polytope is

$$[[10, 2^+, 10]] \cong G^{4,4,10}$$

of order 400 (see (2.39)), in agreement with the fact that there are 100 vertices while the symmetry group of the vertex figure is the 4-group $\mathfrak{D}_2$ generated by R_1 and R_4. (This description of the symmetry group of the grand antiprism was kindly provided by Professor Norman W. Johnson.)

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In addition to 'Part I', which is

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