Four polytope products:
Join, Fusil, Prism and Meet

Abstract: An *n*-polytope is defined recursively by (*n*-1)-polytope facets, 2 per ridge. A 1-polytope: 2 points, a segment body. A 2-polytope: polygon with vertices and edges, 2 edges/vertex. A 3-polytope: polyhedron, polygonal faces, 2 faces/edge. Polytopes can be characterized by f-vectors, like a *p*-gon's f-vector: (*p*,*p*), *p* vertices and edges. Product polytopes like prisms and dual bipyramids known since Kepler. Self-dual pyramids, skew polytopes also intriguing product forms. This talk presents 4 product operators: join, fusil, prism, meet ($\vee$, +, ×, $\vee$), computing f-vectors via vector products. E.g., cube's f-vector (8,12,6,1) from prism triple product of segments (2,1), coefficients from characteristic polynomial $(2+x)^3$.

Presented: Duluth MAA conference, Saturday September 23, 2023

https://www.northcentral.maa.org/maa-ncs-fall-2023-meeting-1

[Paper in peer review](#) (feedback welcome! – tomruen@gmail.com)

Four Polytope Products: Join, Fusil, Prism, and Meet

Thomas L Ruen

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Abstract: There are four special operators on polytopes – join, fusil, prism, and meet. Joins create pyramid forms, connecting all elements. Fusils creates cross polytopes, also connecting all elements, but excluding bodies. Prisms create hypercubes as Cartesian or rectangular products. Meets also create Cartesian products, but generate skew polytopes by removing body elements. This paper explores lower dimensional examples of these operators and how to work with them. The f-vectors for all 4 operators can be generated by products like coefficients of polynomial products. Detailed k-face elements are computed via product tables and can be shown in Hasse diagrams.

1 Introduction: Four Operators

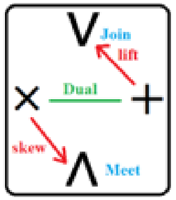


Figure 1

In this paper we use Norman Johnson names [4] with join, ∨, fusil (rhombic "sum"), +, prism (Cartesian or rectangular product), ×. Johnson named fusil from a rhombic shape, and it also contains the verb fuse as the element polytopes can share the same center. Fusils are given a + symbol because vertex counts add, while prism has a × symbol because vertices multiply. Johnson didn't mention specifically the final meet ∧, but join-meet are a natural extension, reminding us of union and intersection.

A 2016 paper [3], described all 4 products and names our meet operator as a *topological product*.

Richard Klitzing [5] name the four operators. The join operator is $\times_{1,1}$, called a pyramid product. The fusil operator is $\times_{1,0}$, called a tegum product. The prism operator, is $\times_{0,1}$. Finally, the meet operator is represented as $\times_{0,0}$, and called a honeycomb product.

Figure 1 shows the join can be seen as a "dimensional lift" of the fusil (+), while prism and fusil are duals, and meet is a skew down-rank construction from the prism.

The operators can be expressed purely abstractly, as orthogonal products of *k*-faces of each polytope. An *i*-element polytope A product with a *j*-element polytope B defines an *i*×*j* matrix of product elements. The join and fusil have joined elements, and the prism and meet have prism elements. These will be described in detailed examples with product tables and Hasse diagrams.

Figure 2 shows example figures for all 4 operators, joining a pentagon and a point, fusing a pentagon and a segment, prism of a pentagon and segment, and a meet of two orthogonal pentagons, projected down from 4-dimensions.

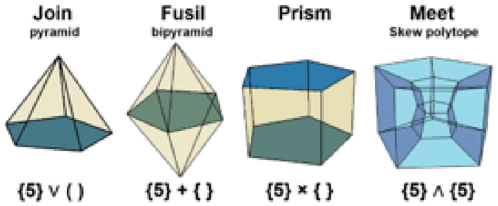


Figure 2

1.1 Polytope Rank

An *n*-[polytope](#) is said to have [rank](#) *n*, bounded by (*n*-1)-faces. A [polygon](#) is rank 2, bounded by 1-faces or edges, [polyhedron](#) rank 3, bounded by 2-faces.

The join operator adds one dimension or rank, while meet subtracts one rank.

- **Join:** Rank(A ∨ B) = Rank(A) + Rank(B) + 1
- **Fusil:** Rank(A + B) = Rank(A) + Rank(B)
- **Prism:** Rank(A × B) = Rank(A) + Rank(B)
- **Meet:** Rank(A ∧ B) = Rank(A) + Rank(B) – 1

1.2 Polytope f-vectors and products

A product polytope's [f-vector](#) can be computed like polynomial products from its elements. The polynomial x^k powers are mapped onto *k*-faces, with term coefficients as each f-vector count element.

The products can be seen in by extended f-vectors. An f-vector lists counts of *k*-faces, *k*=0…*n*-1. A f-vector is extended by a -1-face element (empty set, or [nullitope](#)) as count 1 for the join and fusil operations. The join and prism operators also include the body as an n-face ([body](#)) as 1.

A [*regular polytope*](#) is uniquely defined by its f-vector counts, while other polytopes must include a list of *k*-face polytope types for completeness.

These extended f-vectors are written with a leading 1 (nullitope), or trailing 1 (body). We can use the operators as (∨, +, ×, ∧), or subscript versions used by Klitzing ($\times_{1,1}$, $\times_{1,0}$, $\times_{0,1}$, $\times_{0,0}$).

- Join ∨ or $\times_{1,1}$: (1,f,1)
- Fusil + or $\times_{1,0}$: (1,f)
- Prism × or $\times_{0,1}$: (f,1)
- Meet ∧ or $\times_{0,0}$: (f)

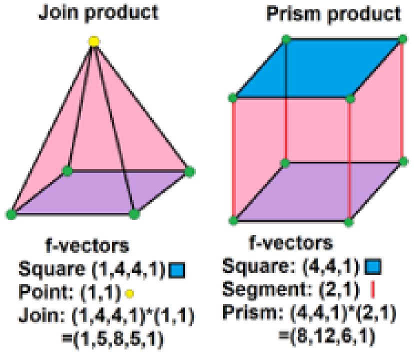


Figure 3

Figure 3 (left) shows a [*square pyramid*](#) as a join of a square base and an offset point.

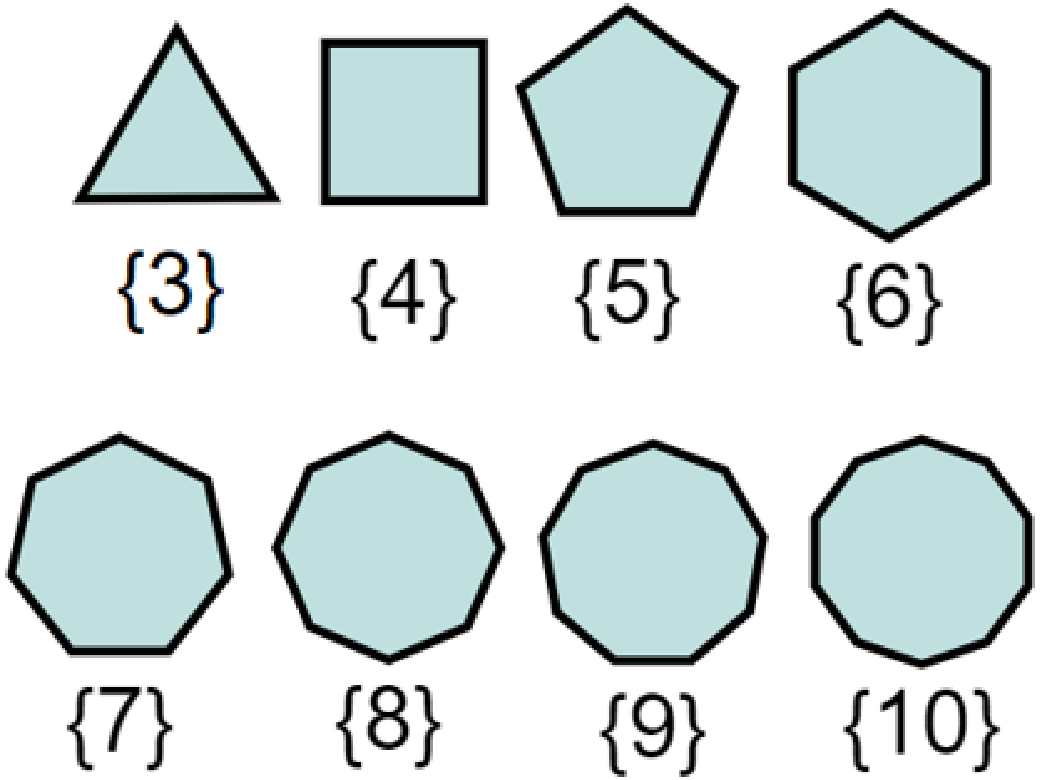
The square base has extended f-vector (1,4,4,1), and a point can be represented as (1,1). Their product can be computed as $(1+4x+4x^2+x^3)(1+x)=1+5x+8x^2+5x^3+x^4$. Then coefficients can be extracted as (1,5,8,5,1). A square pyramid has 5 vertices, 8 edges, and 5 faces.

Figure 3 (right) shows a square prism product. It doesn't include the leading nullitope 1.

My story: Fudging pentagons

1975: Elementary school playground

Ice puddle polygons - Mystical pentagon!

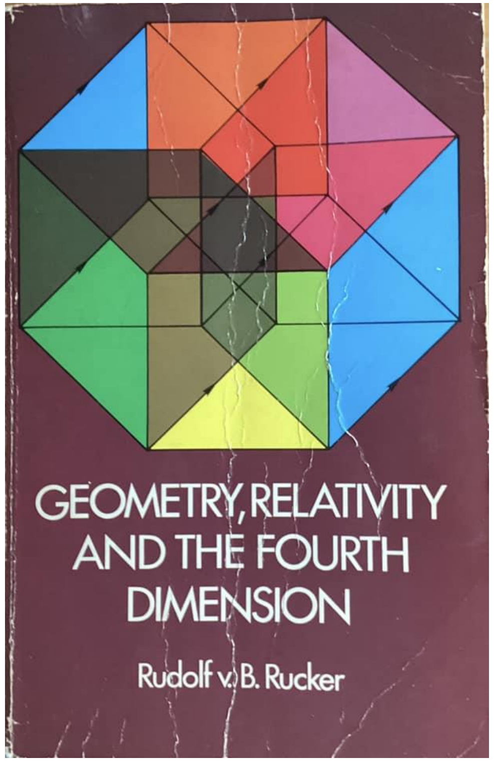


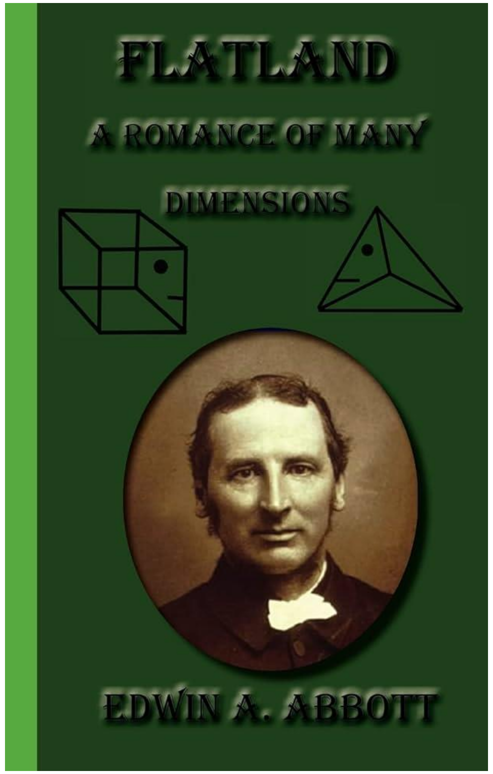
1978 Christmas book

Rudy Rucker (1946) and Edwin Abbott (1808-1882)

(1977) *Geometry, Relativity and the Fourth Dimension*

based on (1884) *Flatland: A Romance of Many Dimensions*

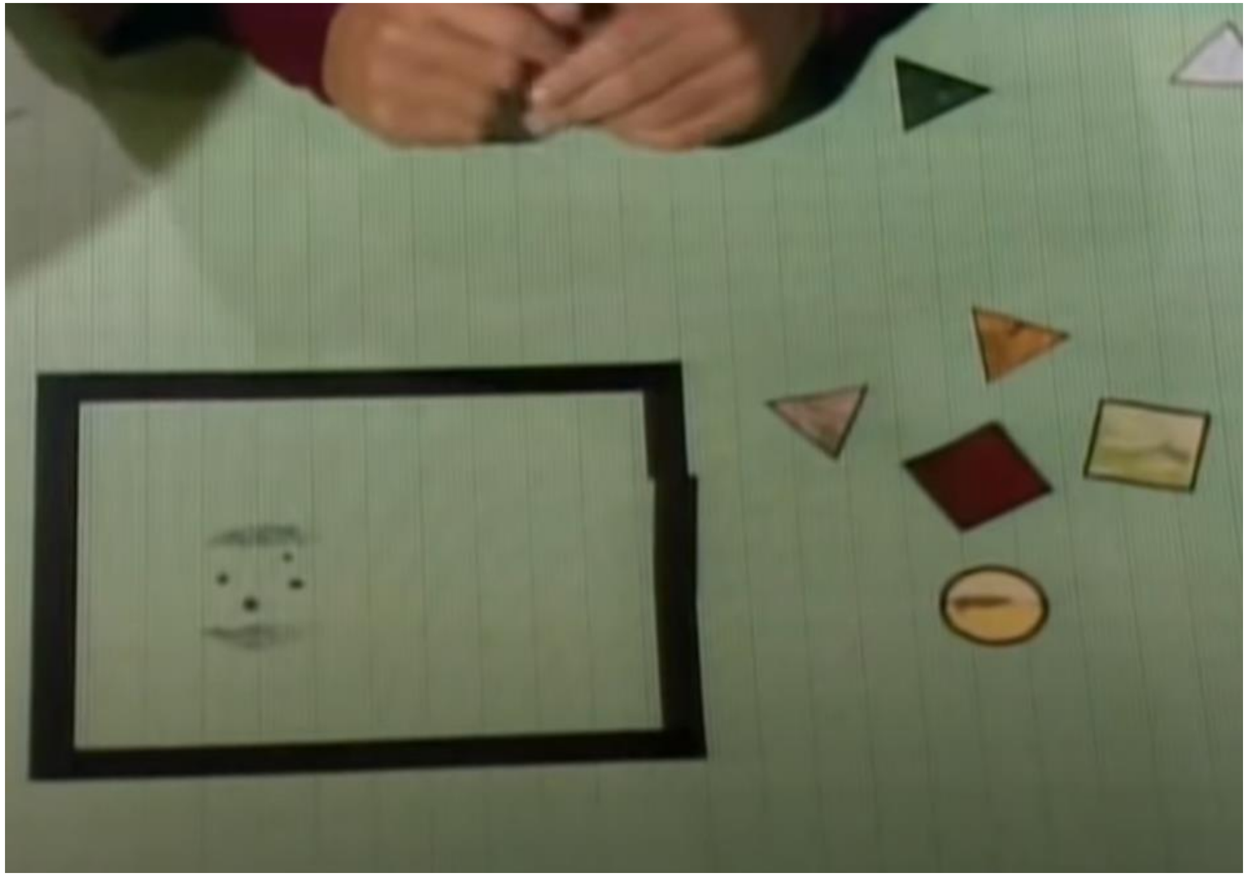




1980 Cosmos (Episode 10: The Edge of Forever)

Lower and higher dimensions

More flatland and shadows of 4th dimension

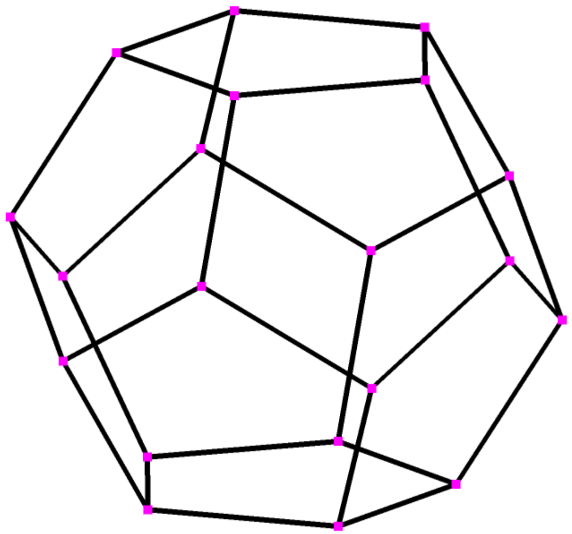


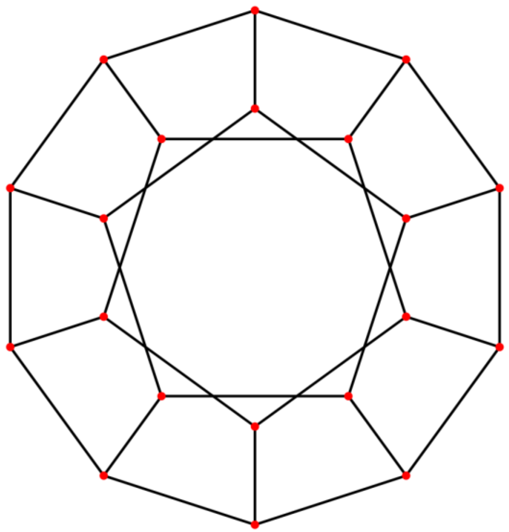
Computer Animation

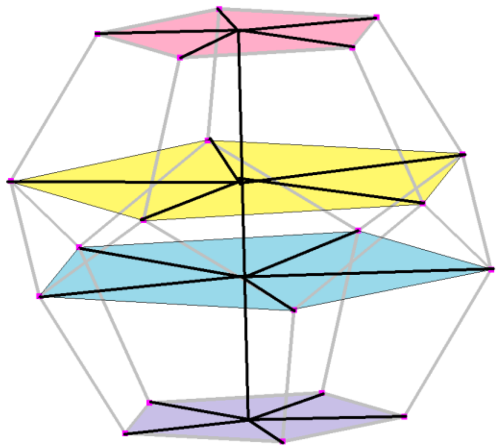
1984: 10th grade geometry class

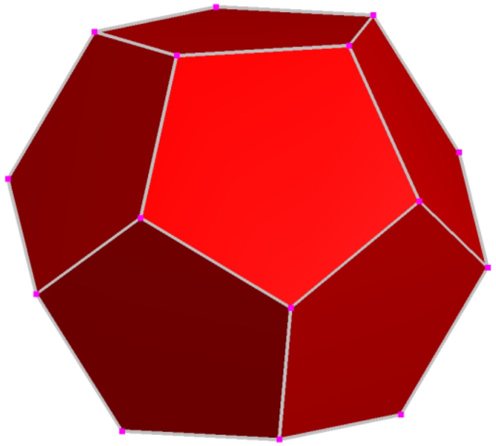
Apple II+ BASIC & assembly

Dodecahedron as 4 pentagons!







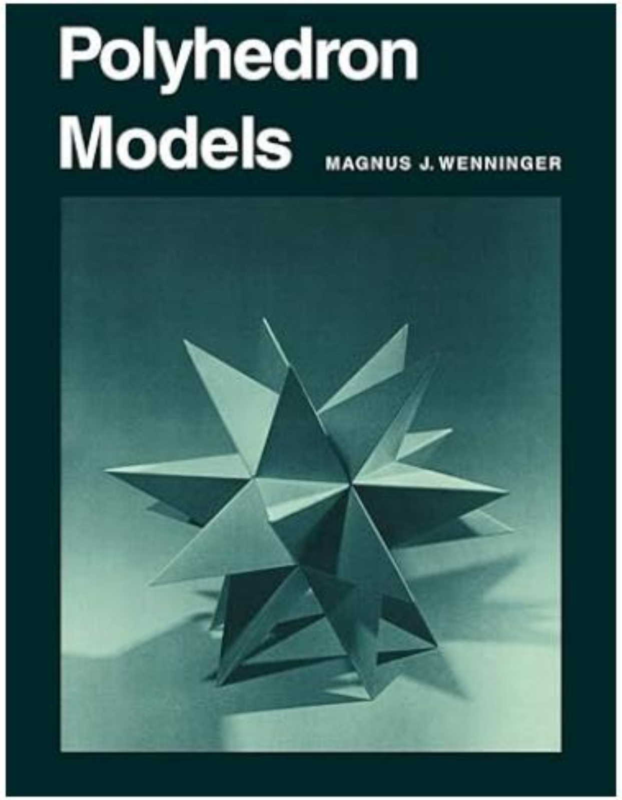


1986 High School wheel-throwing
1990 College ceramics – hand-building

Dodecahedra and more – solids and pinched slab


1990 Saint Cloud Mathematics conference
I met Father Magnus J. Wenninger 1919-2017

Monk of Saint John's Abbey Collegeville, Minnesota (1945)
Mathematics teacher at Saint Augustine's College from 1946 to 1971
Polyhedron Models (1974) – 75 uniform star polyhedra, named by Norman Johnson



Geometry Center

University of Minnesota (~1990-1998)

June 1991 two-week class (HS, undergrads, teachers!)

Geometry and the Imagination

John H Conway, Peter Doyle, Jane Gilman, and Bill Thurston

John H. Conway
(1937-2020)

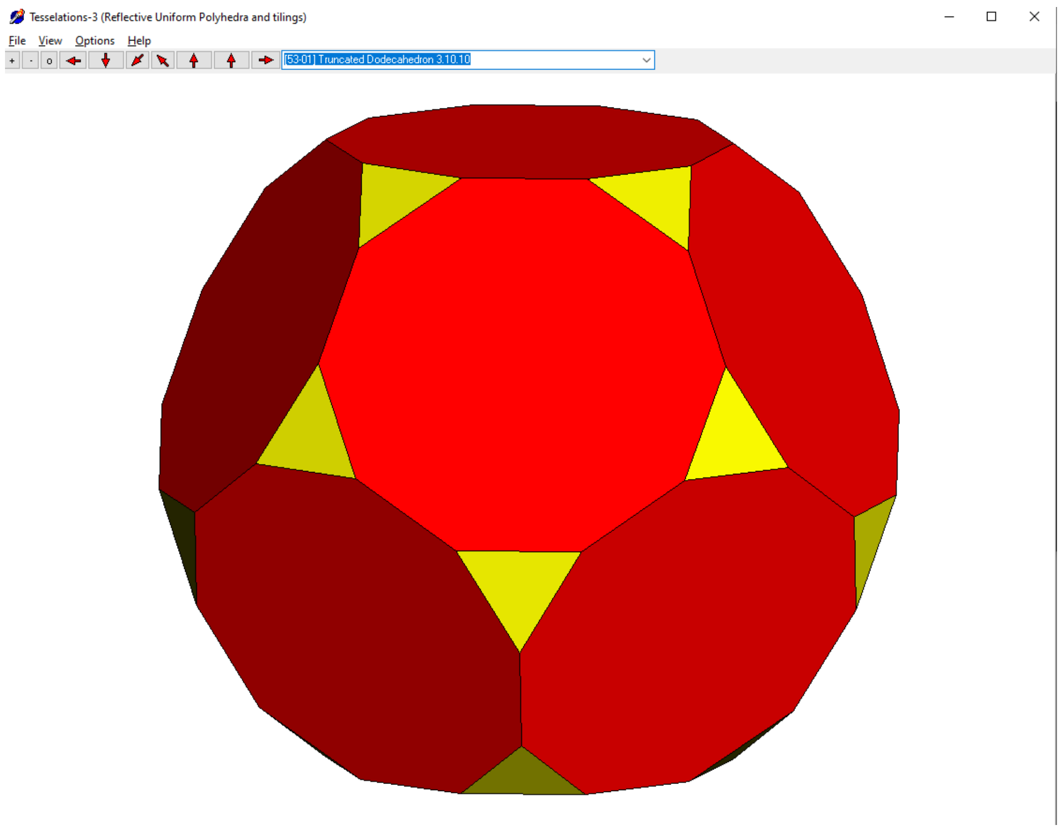
Peter Doyle

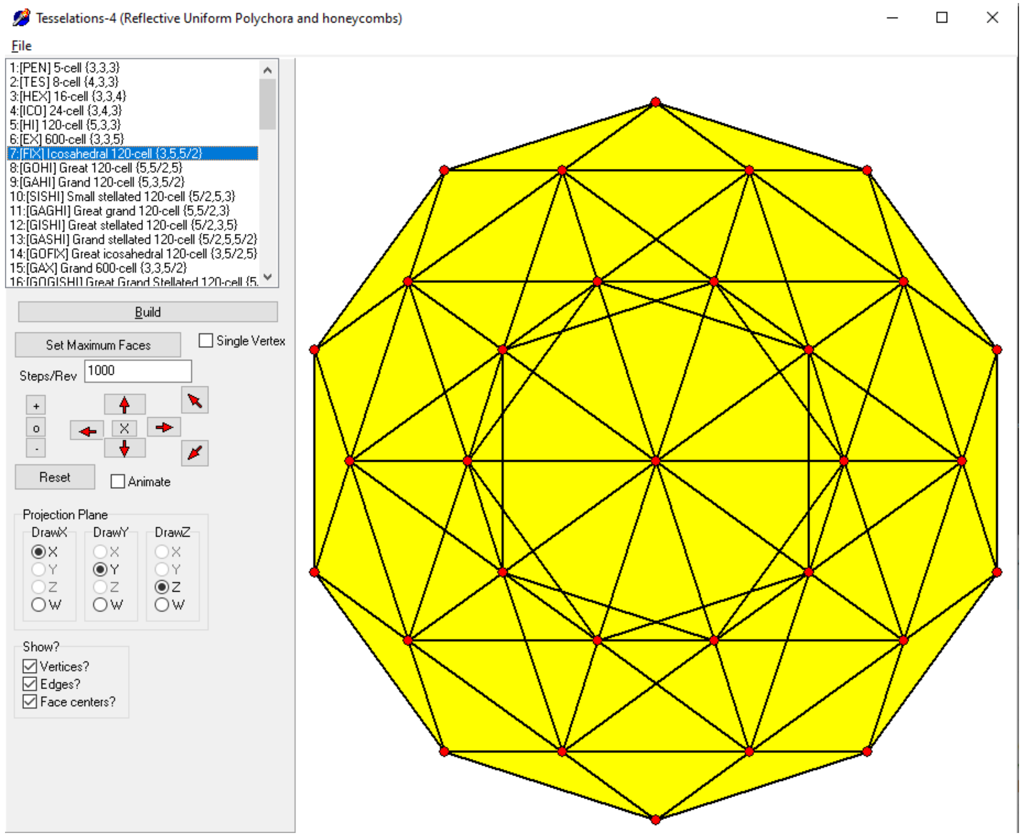
Jane Gilman

Bill Thurston
(1946-2012)

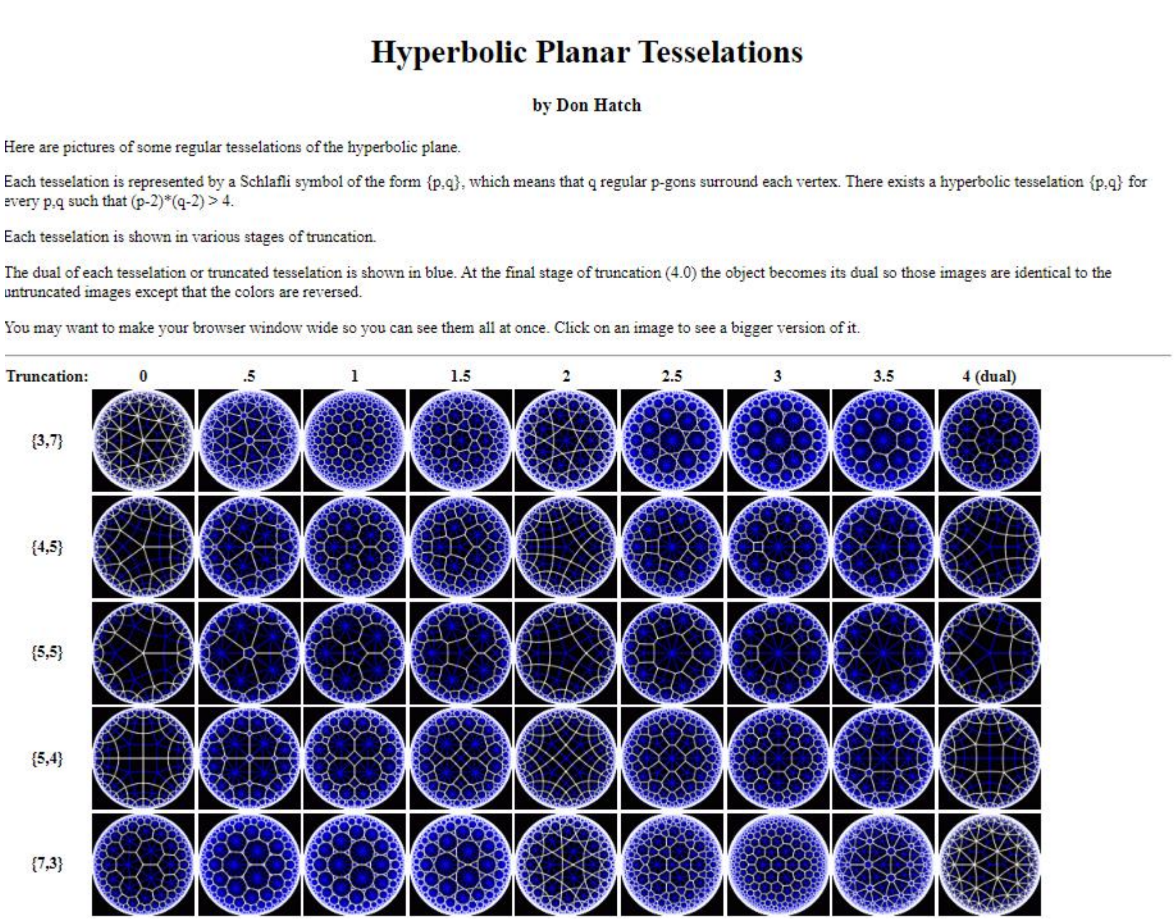
(1997-1999) Delphi Pascal

Polyhedron and 4-polytope building by vertex figure reflection





2003 Hyperbolic tilings – “Impossible” polyhedra



Hyperbolic Planar Tesselations

by Don Hatch

Here are pictures of some regular tesselations of the hyperbolic plane.

Each tesselation is represented by a Schlafli symbol of the form {p,q}, which means that q regular p-gons surround each vertex. There exists a hyperbolic tesselation {p,q} for every p,q such that (p-2)*(q-2) > 4.

Each tesselation is shown in various stages of truncation.

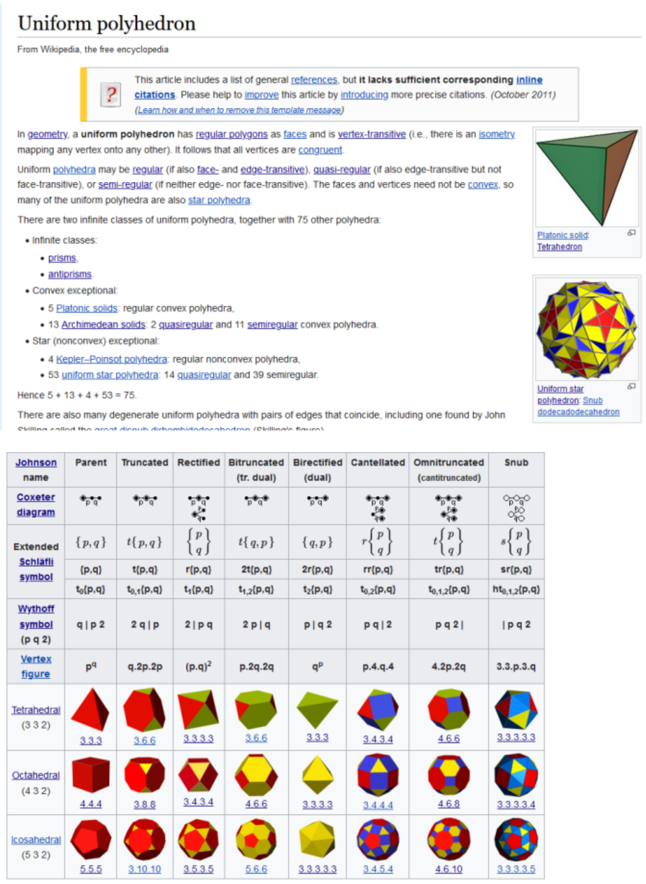
The dual of each tesselation or truncated tesselation is shown in blue. At the final stage of truncation (4.0) the object becomes its dual so those images are identical to the untruncated images except that the colors are reversed.

You may want to make your browser window wide so you can see them all at once. Click on an image to see a bigger version of it.

Truncation:	0	.5	1	1.5	2	2.5	3	3.5	4 (dual)
{3,7}									
{4,5}									
{5,5}									
{5,4}									
{7,3}									

Wikipedia (2004-present)

Uniform polyhedra and 4-polytopes, hyperbolic tilings, honeycombs



Uniform polyhedron

From Wikipedia, the free encyclopedia

This article includes a list of general references, but it lacks sufficient corresponding inline citations. Please help to improve this article by introducing more precise citations. (October 2011) (Learn how and when to remove this template message)

In geometry, a uniform polyhedron has regular polygons as faces and is vertex-transitive (i.e., there is an isometry mapping any vertex onto any other). It follows that all vertices are congruent.

Uniform polyhedra may be regular (if also face- and edge-transitive), quasi-regular (if also edge-transitive but not face-transitive), or semi-regular (if neither edge- nor face-transitive). The faces and vertices need not be convex, so many of the uniform polyhedra are also star polyhedra.

There are two infinite classes of uniform polyhedra, together with 75 other polyhedra:

- Infinite classes:
  - prisms,
  - antiprisms.
- Convex exceptional:
  - 5 Platonic solids: regular convex polyhedra,
  - 13 Archimedean solids: 2 quasiregular and 11 semiregular convex polyhedra.
- Star (nonconvex) exceptional:
  - 4 Kepler–Poinsot polyhedra: regular nonconvex polyhedra,
  - 53 uniform star polyhedra: 14 quasiregular and 39 semiregular.

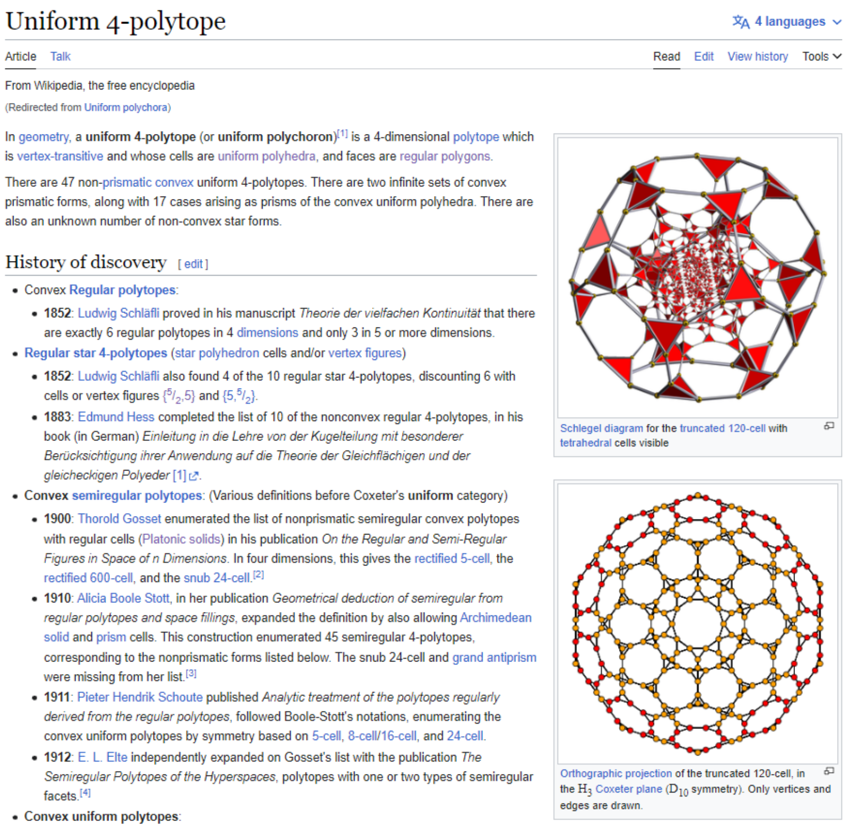
Hence 5 + 13 + 4 + 53 = 75.

There are also many degenerate uniform polyhedra with pairs of edges that coincide, including one found by John

Platonic solid: Tetrahedron

Uniform star polyhedron: Snub dodecadodecahedron

Johnson name	Parent	Truncated	Rectified	Bitruncated (tr. dual)	Birectified (dual)	Cantellated	Omnitruncated (cantitruncated)	Snub
Coxeter diagram								
Extended Schläfli symbol	$\{p,q\}$	$t\{p,q\}$	$\left\{\begin{smallmatrix}p\\q\end{smallmatrix}\right\}$	$t\{q,p\}$	$\{q,p\}$	$r\left\{\begin{smallmatrix}p\\q\end{smallmatrix}\right\}$	$t\left\{\begin{smallmatrix}p\\q\end{smallmatrix}\right\}$	$s\left\{\begin{smallmatrix}p\\q\end{smallmatrix}\right\}$
	{p,q}	t{p,q}	r{p,q}	2t{p,q}	2r{p,q}	rr{p,q}	tr{p,q}	sr{p,q}
	$t_0\{p,q\}$	$t_{0,1}\{p,q\}$	$t_1\{p,q\}$	$t_{1,2}\{p,q\}$	$t_2\{p,q\}$	$t_{0,2}\{p,q\}$	$t_{0,1,2}\{p,q\}$	$ht_{0,1,2}\{p,q\}$
Wythoff symbol (p q 2)	q | p 2	2 q | p	2 | p q	2 p | q	p | q 2	p q | 2	p q 2 |	| p q 2
Vertex figure	p^q	q.2p.2p	$(p.q)^2$	p.2q.2q	q^p	p.4.q.4	4.2p.2q	3.3.p.3.q
Tetrahedral (3 3 2)	3.3.3	3.6.6	3.3.3.3	3.6.6	3.3.3	3.4.3.4	4.6.6	3.3.3.3.3
Octahedral (4 3 2)	4.4.4	3.8.8	3.4.3.4	4.6.6	3.3.3.3	3.4.4.4	4.6.8	3.3.3.3.4
Icosahedral (5 3 2)	5.5.5	3.10.10	3.5.3.5	5.6.6	3.3.3.3.3	3.4.5.4	4.6.10	3.3.3.3.5



Uniform 4-polytope

4 languages

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From Wikipedia, the free encyclopedia

(Redirected from Uniform polychora)

In geometry, a uniform 4-polytope (or uniform polychoron)[1] is a 4-dimensional polytope which is vertex-transitive and whose cells are uniform polyhedra, and faces are regular polygons.

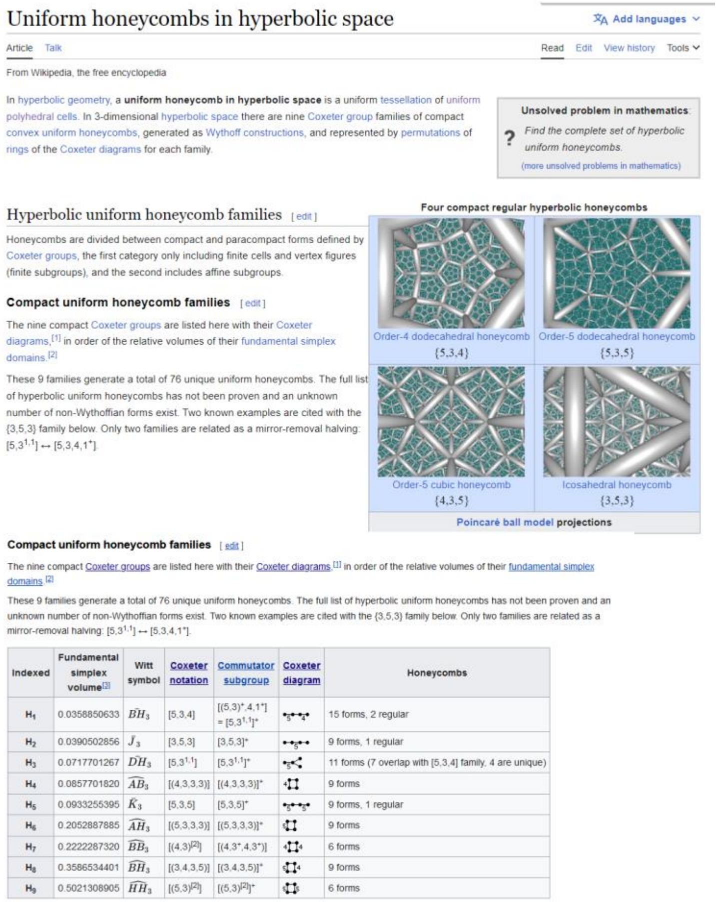
There are 47 non-prismatic convex uniform 4-polytopes. There are two infinite sets of convex prismatic forms, along with 17 cases arising as prisms of the convex uniform polyhedra. There are also an unknown number of non-convex star forms.

History of discovery [edit]

- Convex Regular polytopes:
  - 1852: Ludwig Schläfli proved in his manuscript *Theorie der vielfachen Kontinuität* that there are exactly 6 regular polytopes in 4 dimensions and only 3 in 5 or more dimensions.
- Regular star 4-polytopes (star polyhedron cells and/or vertex figures)
  - 1852: Ludwig Schläfli also found 4 of the 10 regular star 4-polytopes, discounting 6 with cells or vertex figures $\{^5/_2,5\}$ and $\{5,^5/_2\}$.
  - 1883: Edmund Hess completed the list of 10 of the nonconvex regular 4-polytopes, in his book (in German) *Einleitung in die Lehre von der Kugelteilung mit besonderer Berücksichtigung ihrer Anwendung auf die Theorie der Gleichflächigen und der gleicheckigen Polyeder* [1]
- Convex semiregular polytopes: (Various definitions before Coxeter's uniform category)
  - 1900: Thorold Gosset enumerated the list of nonprismatic semiregular convex polytopes with regular cells (Platonic solids) in his publication *On the Regular and Semi-Regular Figures in Space of n Dimensions*. In four dimensions, this gives the rectified 5-cell, the rectified 600-cell, and the snub 24-cell.[2]
  - 1910: Alicia Boole Stott, in her publication *Geometrical deduction of semiregular from regular polytopes and space fillings*, expanded the definition by also allowing Archimedean solid and prism cells. This construction enumerated 45 semiregular 4-polytopes, corresponding to the nonprismatic forms listed below. The snub 24-cell and grand antiprism were missing from her list.[3]
  - 1911: Pieter Hendrik Schoute published *Analytic treatment of the polytopes regularly derived from the regular polytopes*, followed Boole-Stott's notations, enumerating the convex uniform polytopes by symmetry based on 5-cell, 8-cell/16-cell, and 24-cell.
  - 1912: E. L. Elte independently expanded on Gosset's list with the publication *The Semiregular Polytopes of the Hyperspaces*, polytopes with one or two types of semiregular facets.[4]
- Convex uniform polytopes:

Schlegel diagram for the truncated 120-cell with tetrahedral cells visible

Orthographic projection of the truncated 120-cell, in the H_3 Coxeter plane (D_{10} symmetry). Only vertices and edges are drawn.



Uniform honeycombs in hyperbolic space

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From Wikipedia, the free encyclopedia

In hyperbolic geometry, a uniform honeycomb in hyperbolic space is a uniform tessellation of uniform polyhedral cells. In 3-dimensional hyperbolic space there are nine Coxeter group families of compact convex uniform honeycombs, generated as Wythoff constructions, and represented by permutations of rings of the Coxeter diagrams for each family.

Unsolved problem in mathematics: Find the complete set of hyperbolic uniform honeycombs. (more unsolved problems in mathematics)

Hyperbolic uniform honeycomb families [edit]

Honeycombs are divided between compact and paracompact forms defined by Coxeter groups, the first category only including finite cells and vertex figures (finite subgroups), and the second includes affine subgroups.

Compact uniform honeycomb families [edit]

The nine compact Coxeter groups are listed here with their Coxeter diagrams,[1] in order of the relative volumes of their fundamental simplex domains.[2]

These 9 families generate a total of 76 unique uniform honeycombs. The full list of hyperbolic uniform honeycombs has not been proven and an unknown number of non-Wythoffian forms exist. Two known examples are cited with the {3,5,3} family below. Only two families are related as a mirror-removal halving: $[5,3^{1,1}] \leftrightarrow [5,3,4,1^+]$.

Four compact regular hyperbolic honeycombs

Order-4 dodecahedral honeycomb {5,3,4}

Order-5 dodecahedral honeycomb {5,3,5}

Order-5 cubic honeycomb {4,3,5}

Icosahedral honeycomb {3,5,3}

Poincaré ball model projections

Compact uniform honeycomb families [edit]

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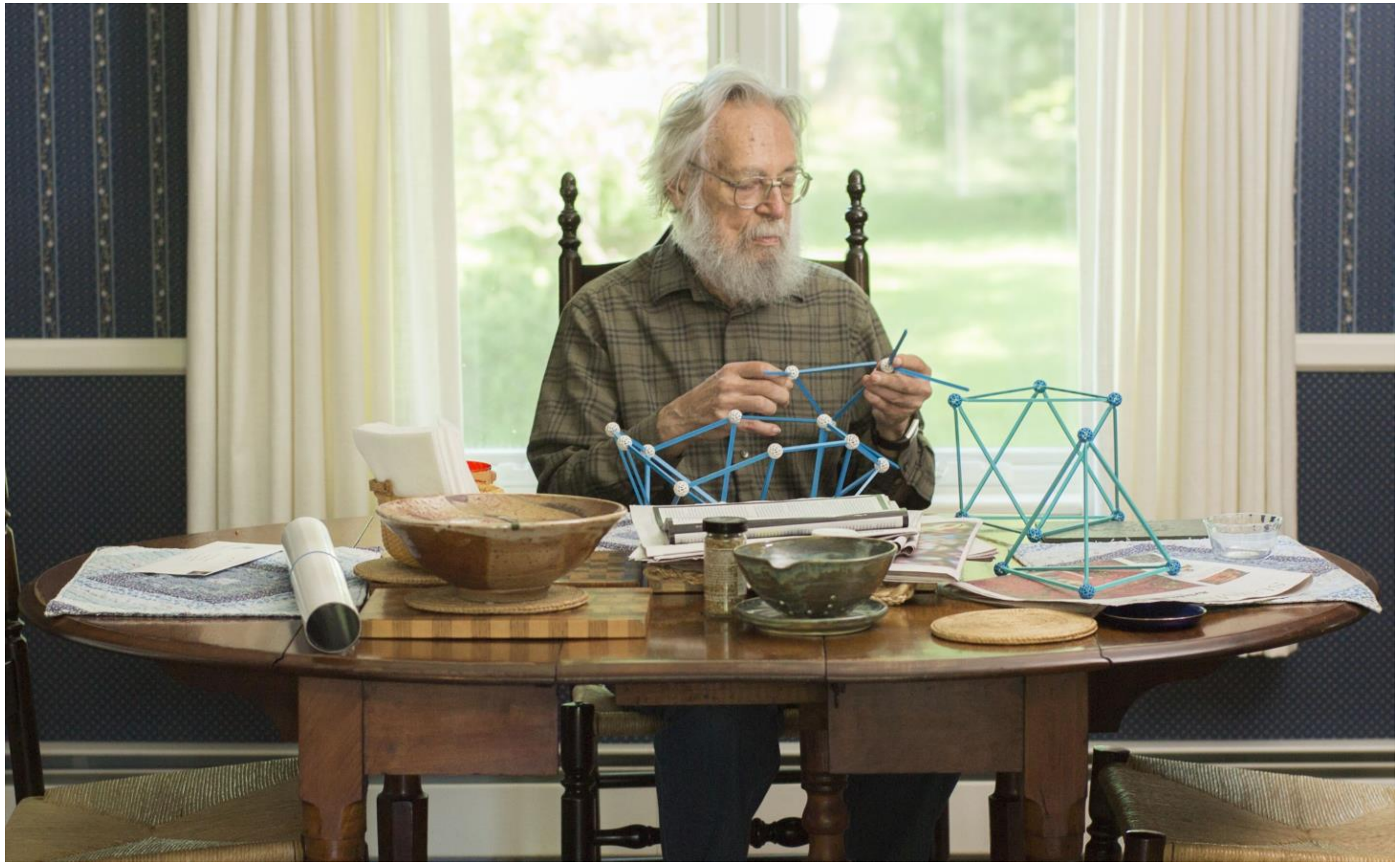
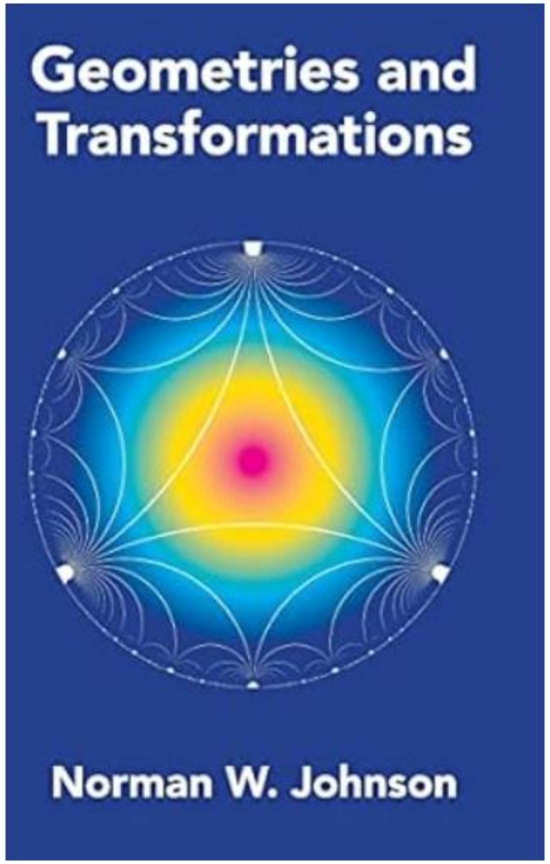
Indexed	Fundamental simplex volume[3]	Witt symbol	Coxeter notation	Commutator subgroup	Coxeter diagram	Honeycombs
H_1	0.0358850633	$\overline{BH}_3$	[5,3,4]	$[(5,3)^+,4,1^+] = [5,3^{1,1}]^+$		15 forms, 2 regular
H_2	0.0390502856	$\overline{J}_3$	[3,5,3]	$[3,5,3]^+$		9 forms, 1 regular
H_3	0.0717701267	$\overline{DH}_3$	$[5,3^{1,1}]$	$[5,3^{1,1}]^+$		11 forms (7 overlap with [5,3,4] family, 4 are unique)
H_4	0.0857701820	$\widehat{AB}_3$	[(4,3,3,3)]	$[(4,3,3,3)]^+$		9 forms
H_5	0.0933255395	$\overline{K}_3$	[5,3,5]	$[5,3,5]^+$		9 forms, 1 regular
H_6	0.2052887885	$\widehat{AH}_3$	[(5,3,3,3)]	$[(5,3,3,3)]^+$		9 forms
H_7	0.2222287320	$\widehat{BB}_3$	$[(4,3)^{[2]}]$	$[(4,3^+,4,3^+)]$		6 forms
H_8	0.3586534401	$\widehat{BH}_3$	[(3,4,3,5)]	$[(3,4,3,5)]^+$		9 forms
H_9	0.5021308905	$\widehat{HH}_3$	$[(5,3)^{[2]}]$	$[(5,3)^{[2]}]^+$		6 forms

Norman W. Johnson 1930-2017

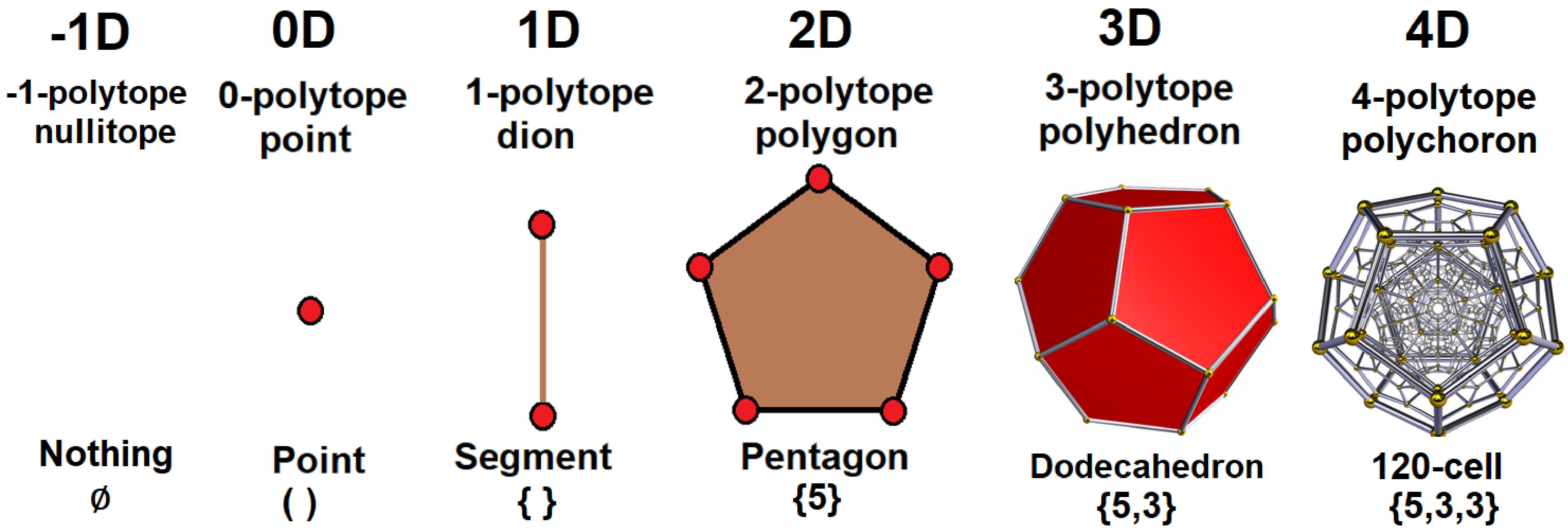
1966 PhD: The Theory of Uniform Polytopes and Honeycombs, under H. S. M. Coxeter
1966: Enumerated, named 92 Convex polyhedra with regular races "Johnson solids"
1998-2017 "Polylist" private forum with George Olshevsky, Jonathan Bowers, John H. Conway, Magnus Wenninger, Richard Klitzing, Wendy Kreiger.
2018: "Geometries and Transformations", portion of unpublished "Uniform polytopes"





What is a polytope?

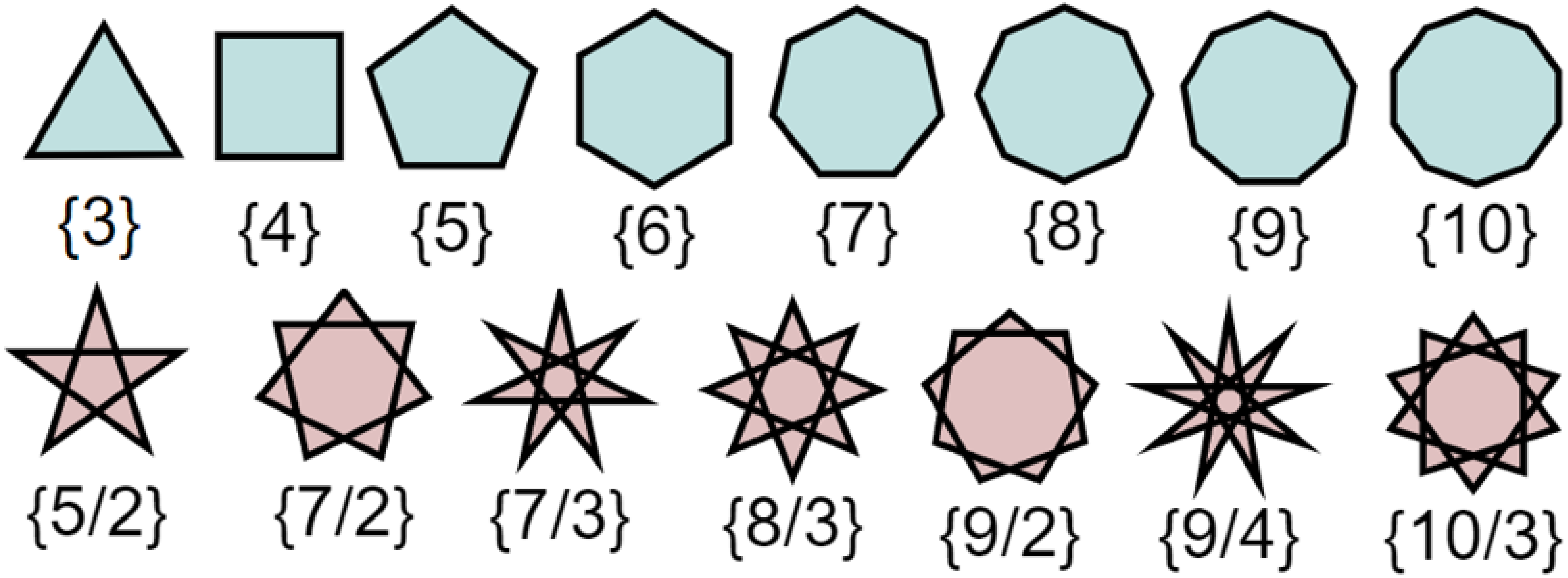
A connection of flat geometric elements.
An *n*-polytope is bounded by (*n*-1)-polytopes called facets (or sides).
A regular polytope has all identical vertices, edges, faces, etc.
The interior of a polytope is called the body.



Schläfli symbol (regular polytopes)

Swiss mathematician Ludwig Schläfli (1814-1895)

* A polygon or p-gon has *p* vertices, *p* edges, 2 edges/vertex.
* A convex regular n-gon has symbol {*p*}, regular star {*p*/*q*}
* Each vertex has 2 edges (manifold condition

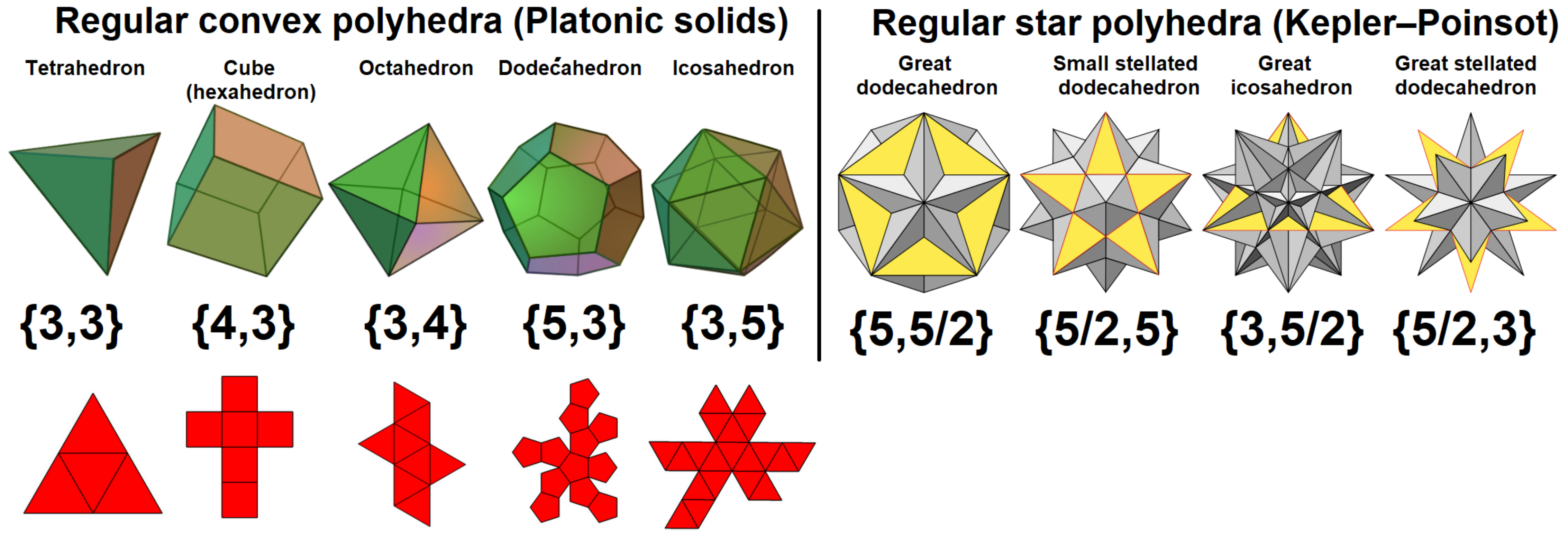


5 regular (Platonic) polyhedra $\{p, q\}$

4 regular star (Kepler-Poinsot) polyhedra $\{p, 5/2\}$, $\{5/2, p\}$

$\{p\}$ faces and q faces around each vertex

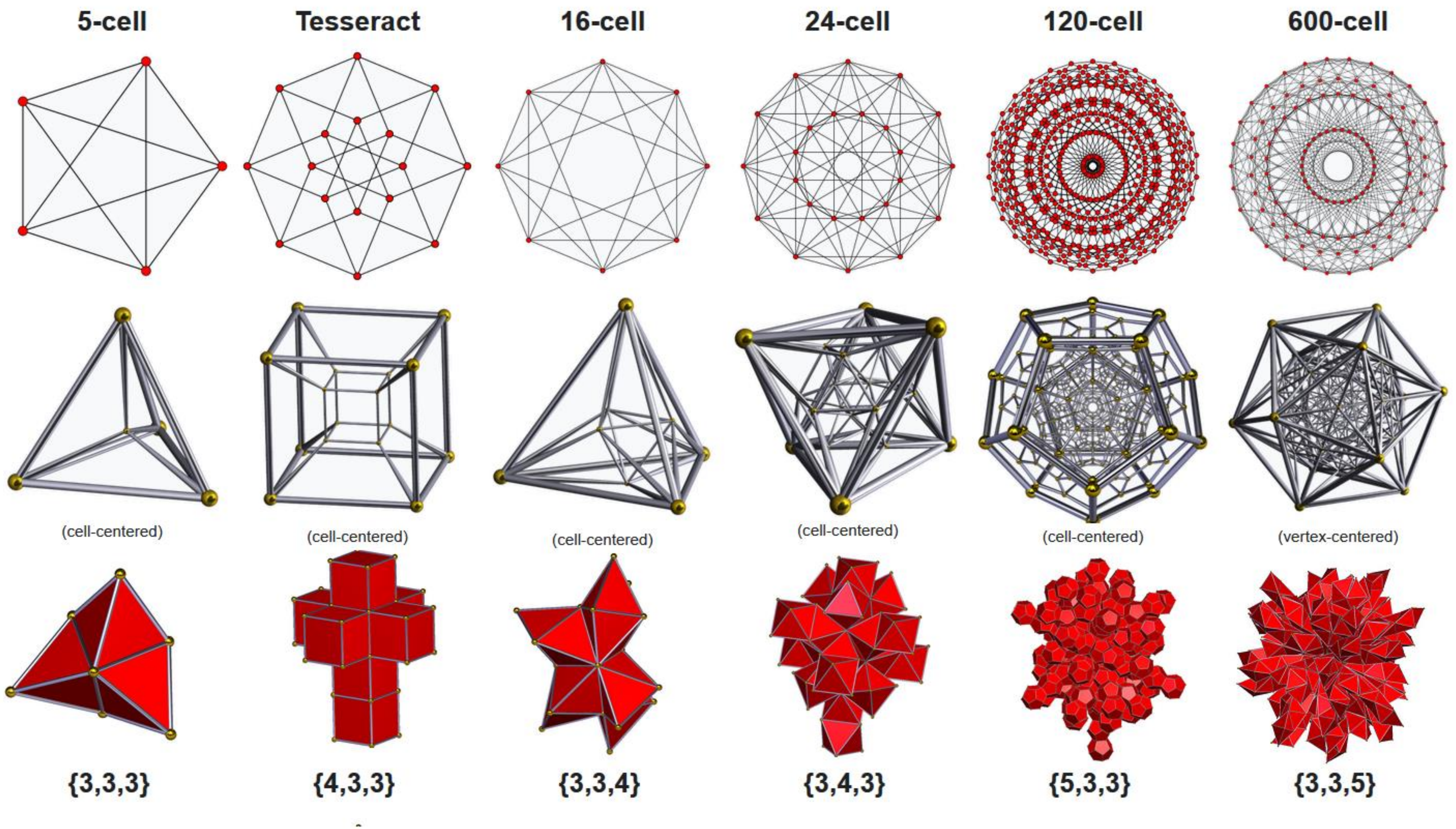
Each edge has 2 faces (manifold condition)



Six regular 4-polytopes: {*p, q, r*}

{*p*, *q*} cells, *r* facets around each edge

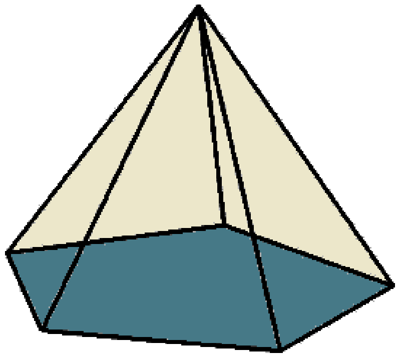
Each face has 2 cells (manifold condition)



Four Polytope Products

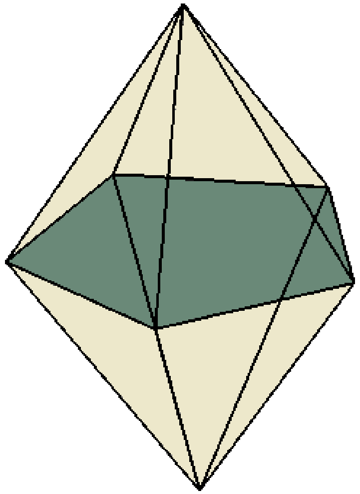
Gleason, I.; Hubard, I., Products of abstract polytopes, 11 Mar 2016
https://arxiv.org/abs/1603.03585

Join
pyramid



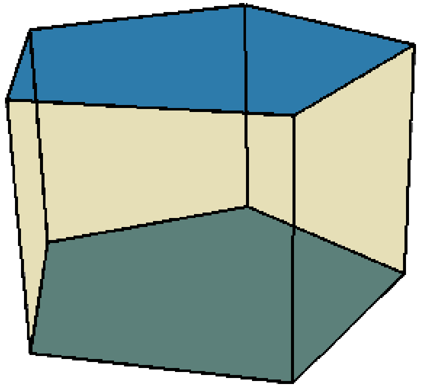
$\{5\} \vee (\,)$

Fusil
bipyramid



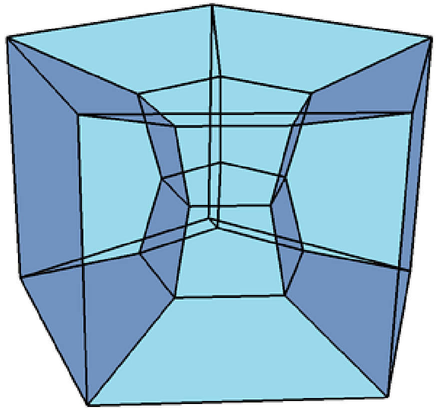
$\{5\} + \{\,\}$

Prism



$\{5\} \times \{\,\}$

Meet
Skew polytope



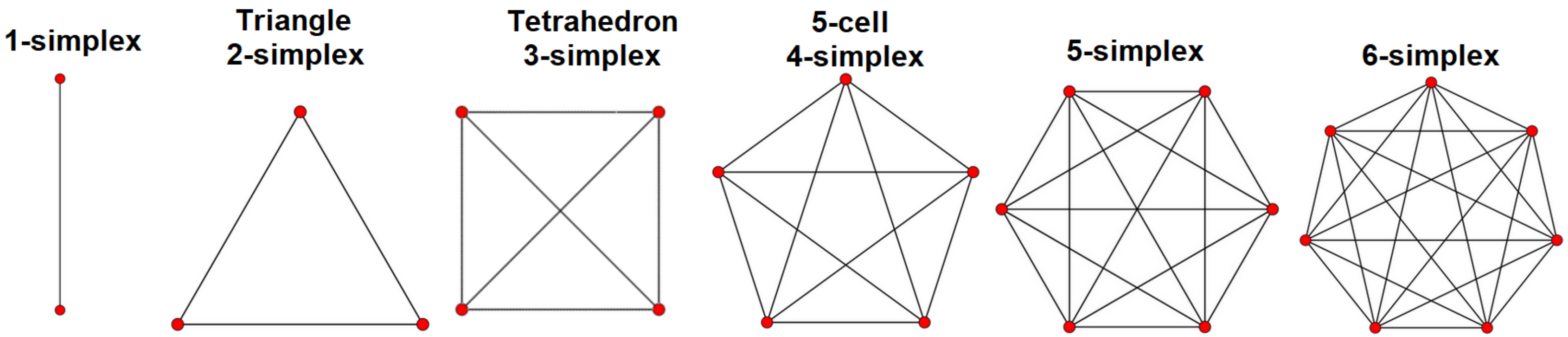
$\{5\} \wedge \{5\}$

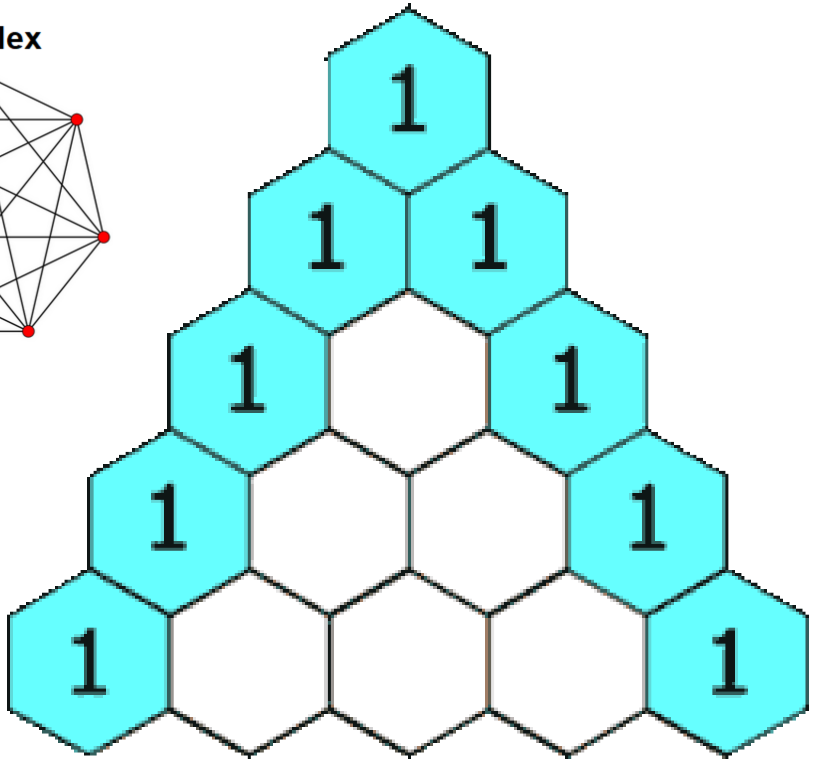
4 regular polytope families in *n*-dimensions

(Each constructed by one of 4 product operators)

Operator	Recursive Product Notation	Schläfli Symbol	Polytope family	Vertices
Join	$(n+1) \cdot (\ )$	$\{3^{n-1}\}$	*n*-simplex	$n+1$
Fusil	$n\ \{\ \}$	$\{3^{n-2},4\}$	*n*-cross polytope	$2n$
Prism	$\{\ \}^n$	$\{4,3^{n-2}\}$	*n-prism*, *n-cube*	2^n
Meet	$\{p\}^{(n)}$ $\{\infty\}^{(n)}$	$\{4,3^{n-2},4 \mid p\}$ $\{4,3^{n-2},4\}$	Regular skew *n*-cubic honeycomb	p^n ∞

Regular n-simplex family: Join (n+1) · ( )   Pascal's triangle





Join (n+1) vertices: extended f-vectors (1 nullitope, *n*+1 vertices, 1 body)

0-simplex: (1,1) – point

1-simplex: $(1,1)^2$ = (1,2,1) – segment

2-simplex: $(1,1)^3$ = (1,3,3,1) - triangle

3-simplex: $(1,1)^4$ = (1,4,6,4,1) - tetrahedron

4-simplex: $(1,1)^5$ = (1,5,10,10,5,1) – 5-cell

5-simplex: $(1,1)^6$ = (1,6,15,20,15,6,1)

6-simplex: $(1,1)^7$ = (1,7,21,35,35,21,7,1)

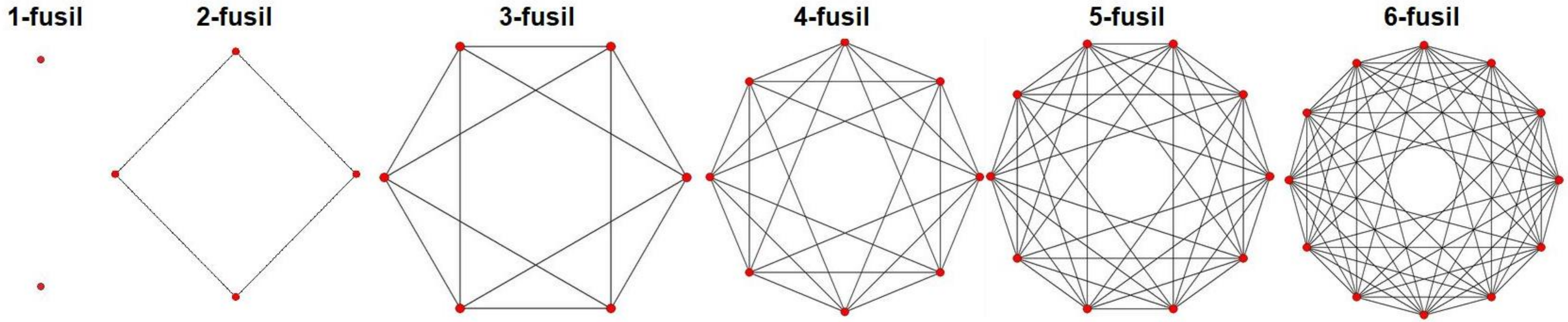
7-simplex: $(1,1)^8$ = (1,8,28,56,70,56,28,8,1)

8-simplex: $(1,1)^9$ = (1,9,36,74,126,74,36,9,1)

Binomial theorem $(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$

Regular n-fusil family: n { }

(Cross polytope, orthoplex)
Direct sum of segments
$2n$ vertices, $2n(n-1)$ edges



n-fusil: extended f-vector: (1 nullitope, $2n$ vertices)

1-fusil: { } : (1,2) *** Segment

2-fusil: 2{ } : $(1,2)^2 = (1,4,4)$ *** Square

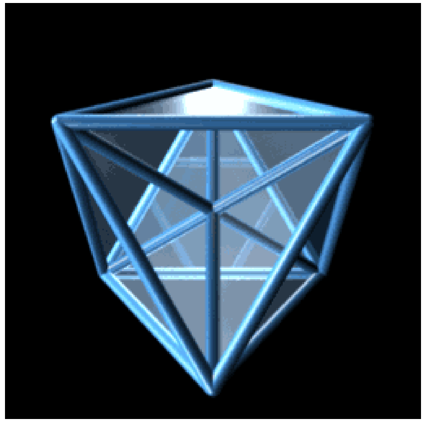
3-fusil: 3{ } : $(1,2)^3 = (1,6,12,8)$ *** Octahedron

4-fusil: 4{ } : $(1,2)^4 = (1,8,24,32,16)$ *** 16-cell

5-fusil: 5{ } : $(1,2)^5 = (1,10,40,80,80,32)$

6-fusil: 6{ } : $(1,2)^6 = (1,12,60,160,240,192,64)$

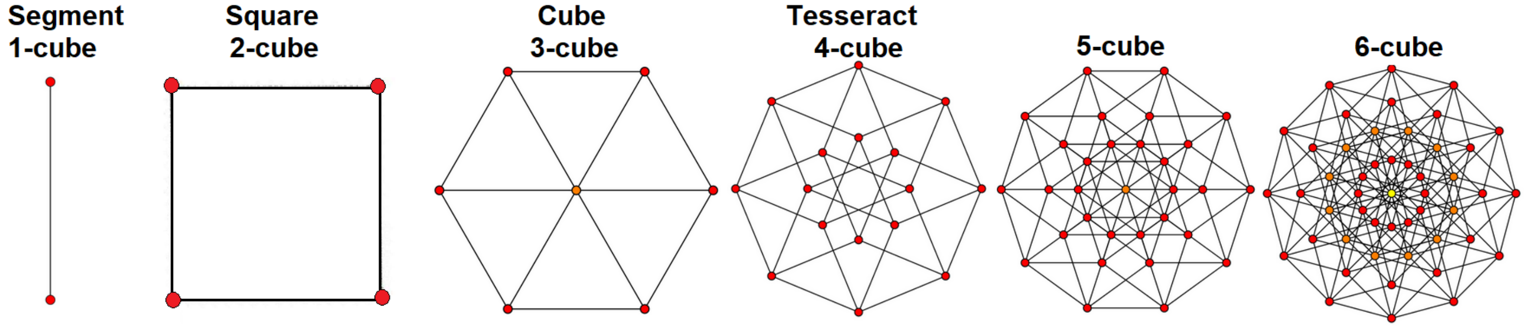
16-cell



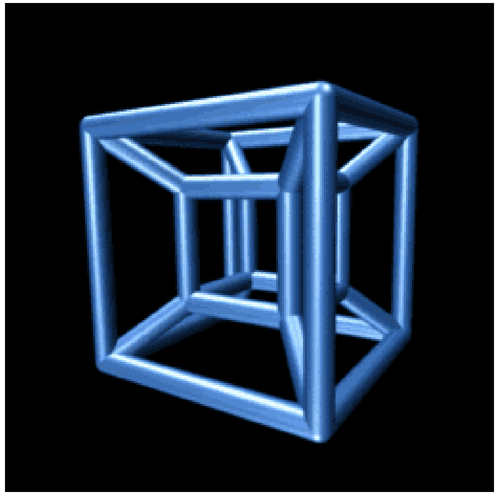
Binomial Theorem $(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$

Regular n-prism family: $\{ \}^n$

(hypercube, *n*-cube, *n*-orthotope)
Cartesian product of segments



Tesseract



n-prism, $\{ \}^n$: extended f-vectors: (2^n vertices, 1 body)

1-prism $\{ \}$: (2,1) *** Segment

2-prism $\{ \}^2$: $(2,1)^2$ = (4,4,1) *** Square, {4}

3-prism $\{ \}^3$: $(2,1)^3$ = (8,12,6,1) *** Cube, {4,3}

4-prism $\{ \}^4$: $(2,1)^4$ = (16,32,24,8,1) *** Tesseract, {4,3,3}

5-prism $\{ \}^5$: $(2,1)^5$ = (32,80,80,40,10,1) *** {4,3,3,3}

6-prism $\{ \}^6$: $(2,1)^6$ = (64,192,240,160,60,12,1) *** {4,3,3,3,3}

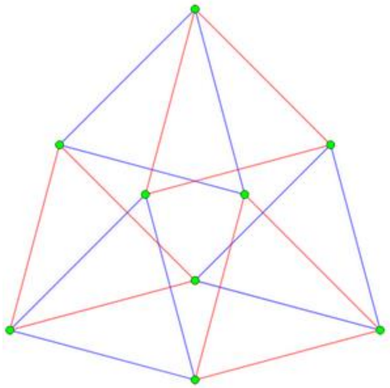
7-prism $\{ \}^7$: $(2,1)^7$ =(128,448,672,570,280,84,14,1) *** {4,3,3,3,3,3}

8-prism $\{ \}^8$: $(2,1)^8$ = (256,1024,1792,1792,1120,448,112,16,1) *** {4,3,3,3,3,3,3}

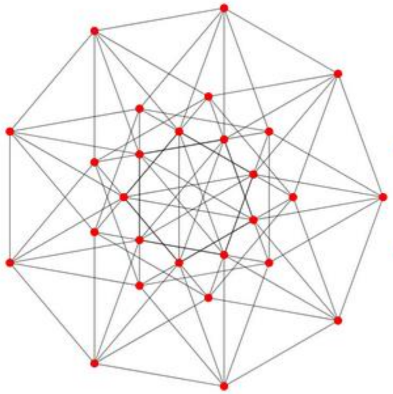
Regular n-meet: $\{p\}^{(n)}$

Skew polytopes, vertices and edges as $\{p\}^n$ n-prism

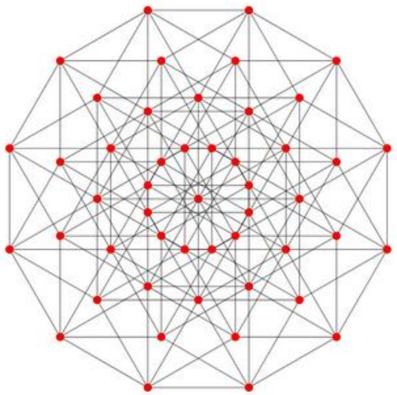
$\{3\}^{(2)}$



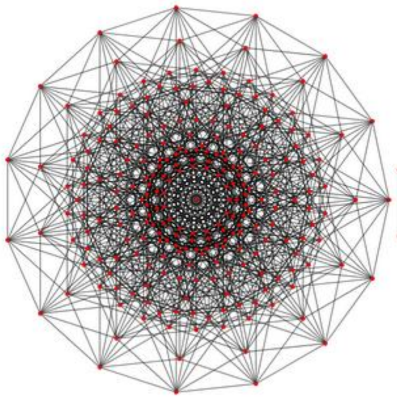
$\{3\}^{(3)}$



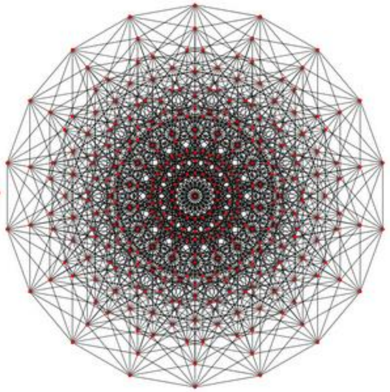
$\{3\}^{(4)}$



$\{3\}^{(5)}$



$\{3\}^{(6)}$



Example: $\{3\}^{(n)}$, triangular n-meets: (3^n vertices on an n-torus surface of a n-sphere)

1-meet $\{3\}^{(1)}$: $(3,3)^1 = 3(1,1) = (3,3)$ *** Triangle

2-meet $\{3\}^{(2)}$: $(3,3)^2 = 3^2(1,2,1) = (9,18,19)$ *** {4,4 | 3}, partial square tiling, 2-torus , 3×3 square net

3-meet $\{3\}^{(3)}$: $(3,3)^3 = 3^3(1,3,3,1) = (27,81,81,27)$ *** {4,3,4 | 3}, partial cubic honeycomb, 3-torus , 3×3×3 cubic net

4-meet $\{3\}^{(4)}$: $(3,3)^4 = 3^4(1,4,6,4,1) =(81,324,486,324,81)$ *** {4,3,3,4 | 3}, partial tesseratic honeycomb, 4-torus, 3×3×3×3 net

5-meet $\{3\}^{(5)}$: $(3,3)^5 = 3^5(1,5,10,10,5,1) = (243,1215,2430,2430,1215,243)$ *** {4,3,3,3,4 | 3}, 5-torus

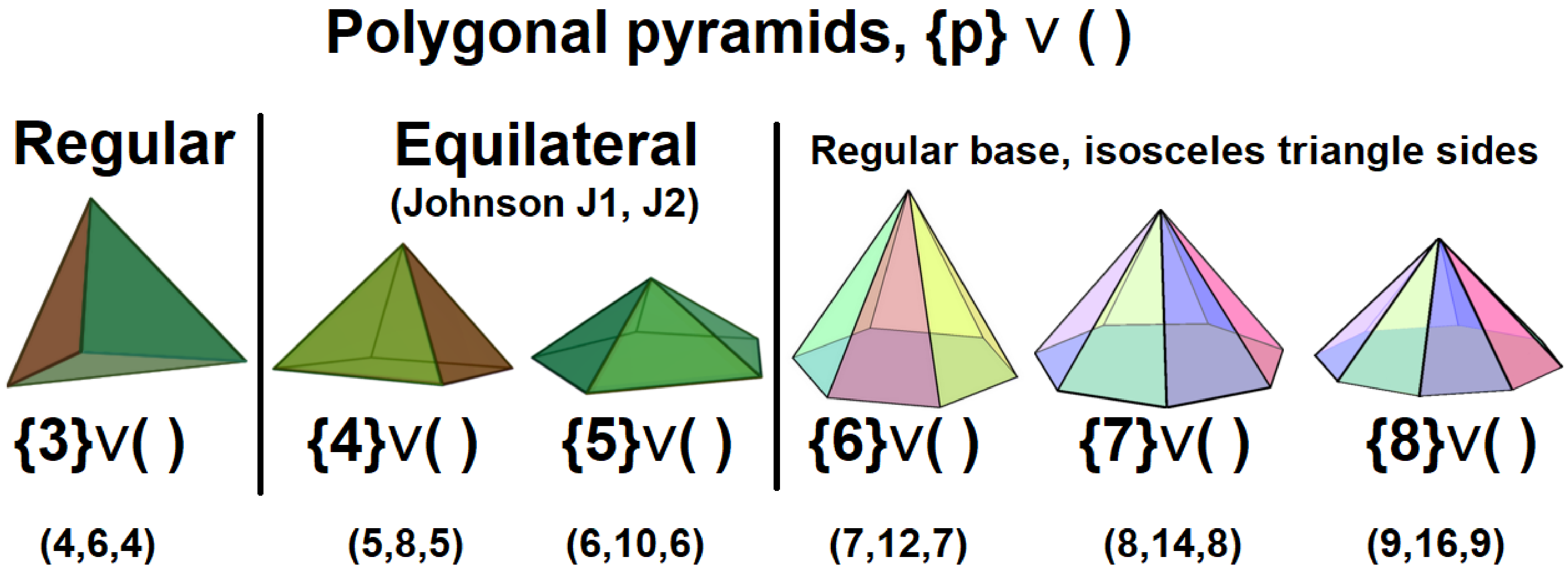
6-meet $\{3\}^{(6)}$: $(3,3)^6 = 3^6(1,6,15,20,15,6,1) = (729,4374,10935,14580,10935,4374,729)$ *** {4,3,3,3,3,4 | 3}, 6-torus

7-meet $\{3\}^{(7)}$: $(3,3)^7 = 3^7(1,7,21,35,35,21,7,1) = (2187,15309,45927,76545,76545,45927,15309,2187)$

8-meet $\{3\}^{(8)}$: $(3,3)^8 = 3^8(1,8,28,56,70,56,28,8,1) = (6561,52488,183708,367416,459270,367416,183708,52488,6561)$

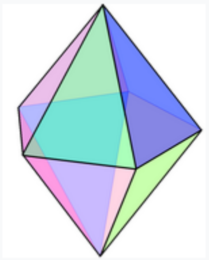
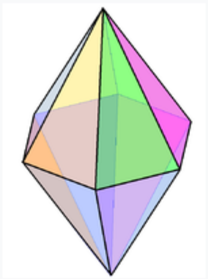
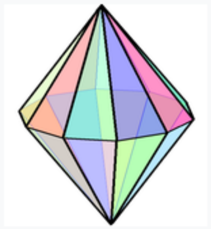
The join operator (∨)

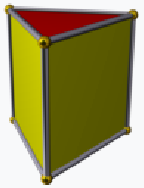
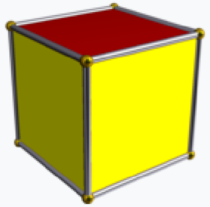
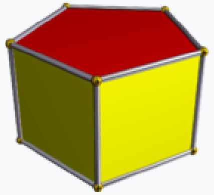
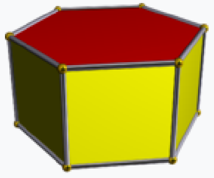
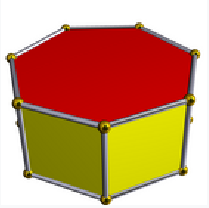
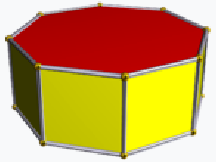
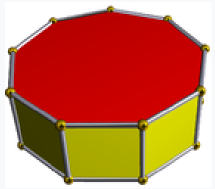
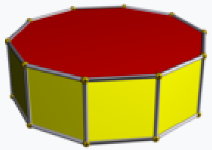
Join attaches two center-offset orthogonal polytopes.
If one polytope is a point, it is pyramid.
A {*p*} ∨ ( ) pyramid has *p*+1 vertices, 2*p* edges, and *p*+1 faces. (self-dual)
Element counts can be given an f-vector (*v*,*e*,*f*) as (*p*+1,2*p*,*p*+1).



Polygonal fusils and prisms (duals)
$\{p\} + \{\ \}$ and $\{p\} \times \{\ \}$

A polygonal fusil is the direct sum of a polygon and a segment.
A polygonal prism is a Cartesian product of a polygon and a segment.
Fusil: f-vector: $(1,p,p)*(1,2)=(1,p+2,3p,2p)$
Prism: f-vector: $(p,p,1)*(2,1)=(2p,3p,2+p,1)$

{3}+{ }	{4}+{ }	{5}+{ }	{6}+{ }	{7}+{ }	{8}+{ }	{9}+{ }	{10}+{ }
							
(1,3,3)*(1,2) (1,5,9,6)	(1,4,4)*(1,2) (1,6,12,8)	(1,5,5)*(1,2) (1,7,15,10)	(1,6,6)*(1,2) (1,8,18,12)	(1,7,7)*(1,2) (1,9,21,14)	(1,8,8)*(1,2) (1,10,24,16)	(1,9,9)*(1,2) (1,11,27,18)	(1,10,10)*(1,2) (1,12,30,20)

{3}×{ }	{4}×{ }	{5}×{ }	{6}×{ }	{7}×{ }	{8}×{ }	{9}×{ }	{10}×{ }
							
(3,3,1)*(2,1) =(6,9,5)	(4,4,1)*(2,1) =(8,12,6)	(5,5,1)*(2,1) =(10,15,7)	(6,6,1)*(2,1) =(12,18,8)	(7,7,1)*(2,1) =(14,21,9)	(8,8,1)*(2,1) =(16,24,10)	(9,9,1)*(2,1) =(18,27,11)	(10,10,1)*(2,1) =(20,30,12)

2-torus – Meet of 2 *p*-gon

Regular skew polyhedron in 4D, $\{p\} \wedge \{p\} = \{p\}^{(2)}$

Example:

$\{20\} \wedge \{20\} = \{20\}^{(2)}$

(20,20)*(20,20)=(400,800,400)

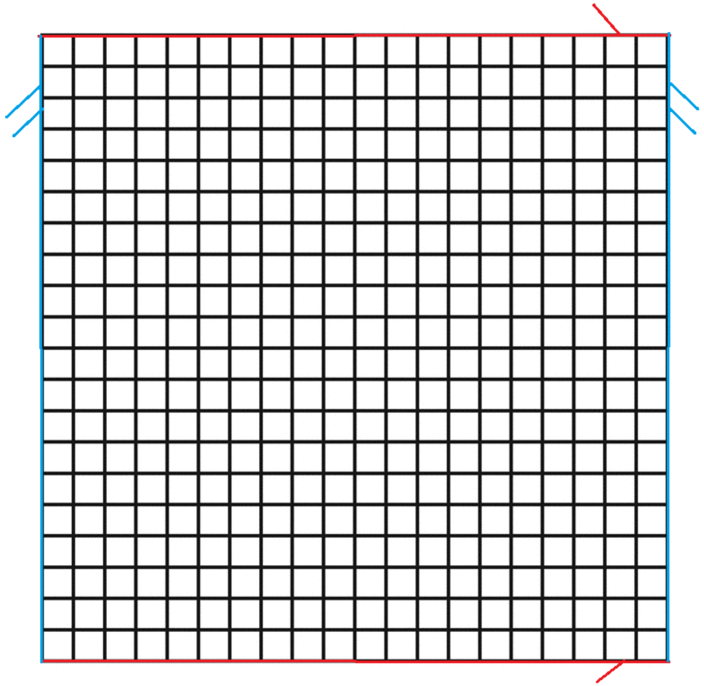
400 vertices, 800 edges

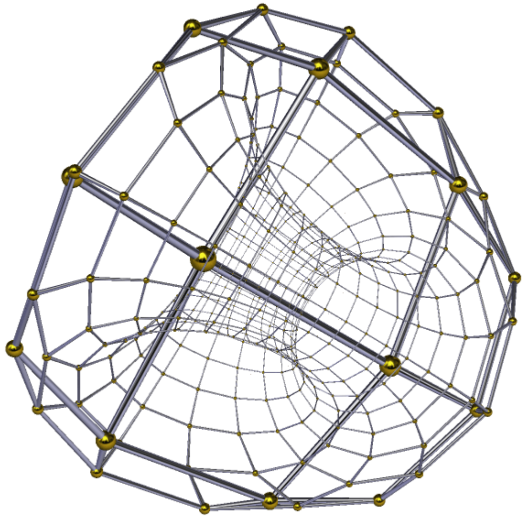
and 400 square faces

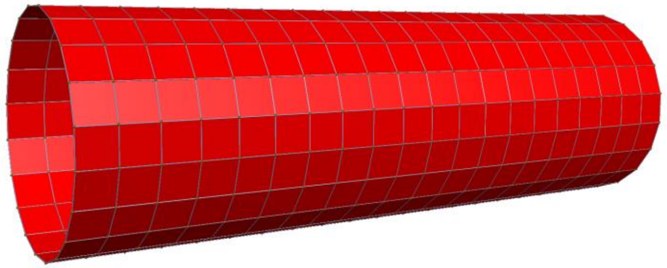
Coxeter name: {4,4 | 20}

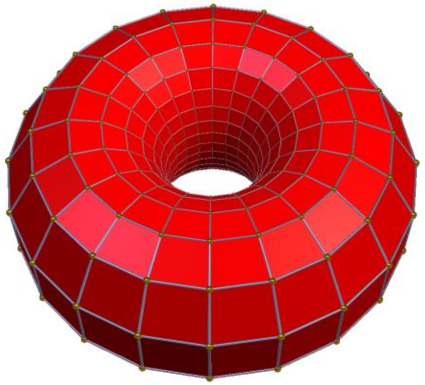
A square tiling {4,4}

repeating every 20 squares



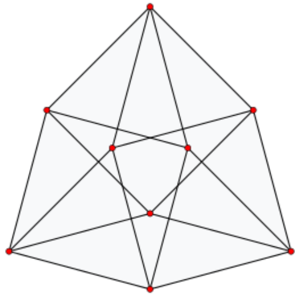
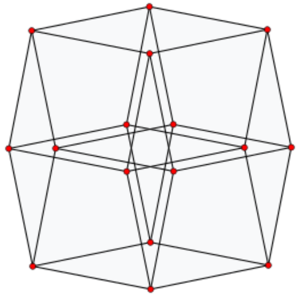
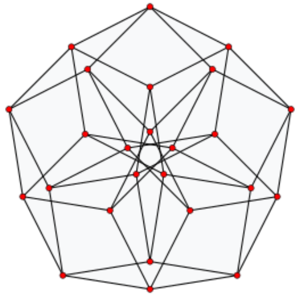
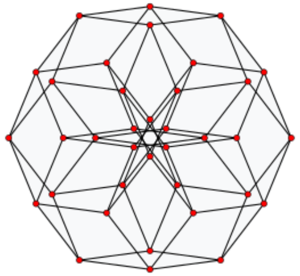
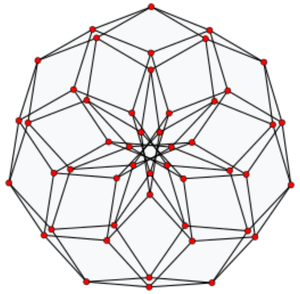
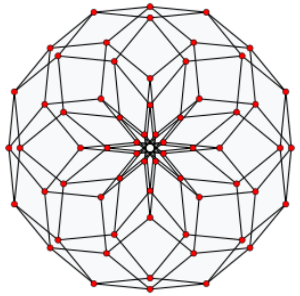
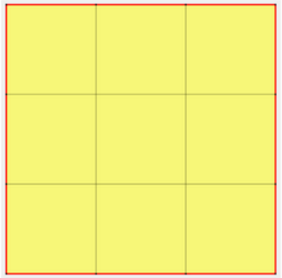
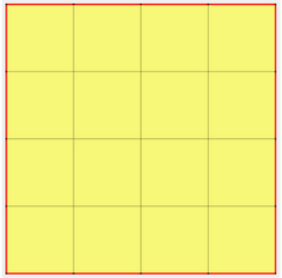
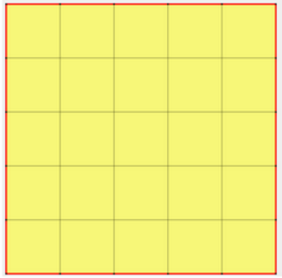
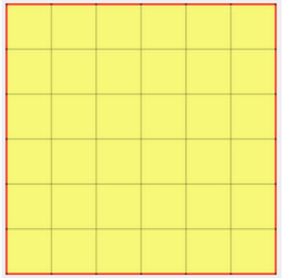
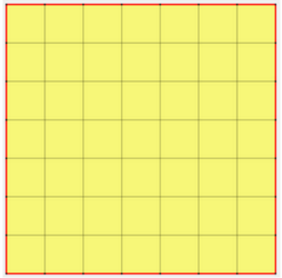
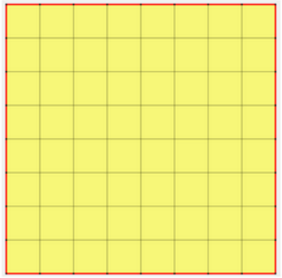






Duoprisms and duomeets

$$\mathrm{Net}(A \wedge B) = \mathrm{Net}(A) \times \mathrm{Net}(B)$$

{3}×{3}	{4}×{4}	{5}×{5}	{6}×{6}	{7}×{7}	{8}×{8}
					
(3,3,1)*(3,3,1) =(9,18,15,6,1)	(4,4,1)*(4,4,1) =(16,32,24,8,1)	(5,5,1)*(5,5,1) =(25,50,35,10,1)	(6,6,1)*(6,6,1) =(36,72,48,12,1)	(7,7,1)*(7,7,1) =(49,98,63,14,1)	(8,8,1)*(8,8,1) =(64,128,80,16,1)
{3}∧{3}	{4}∧{4}	{5}∧{5}	{6}∧{6}	{7}∧{7}	{8}∧{8}
					
(3,3)*(3,3) =(9,18,9)	(4,4)*(4,4) =(16,32,16)	(5,5)*(5,5) =(25,50,25)	(6,6)*(6,6) =(36,72,36)	(7,7)*(7,7) =(49,98,49)	(8,8)*(8,8) =(64,128,64)

Magic math of extended f-vectors explained

An f-vector counts elements.
Example:cube (8,12,6), 8 vertices, 12 edges, 6 faces.
An extended f-vector can include 1 (nullitope) and 1 body:

- Join $\vee$ (**1**,f,**1**) - segment (1,2,1) - 1 null. 2 vertices, 1 body
- Fusil + (**1**,f) - segment (1,2) - 1 null., 2 vertices
- Prism × (f,**1**) - segment (2,1) - 2 vertices, 1 body
- Meet $\wedge$ (f) - segment (2) - 2 vertices

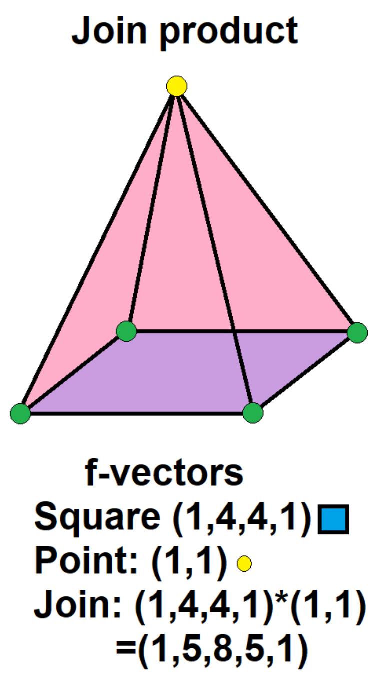
Square pyramid: square (1,4,4,1) and point (1,1)

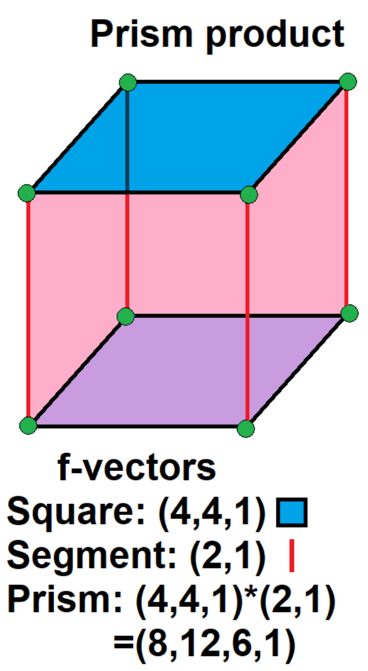
Square prism: square (4,4,1) and segment (2,1)

Products work like polynomials:

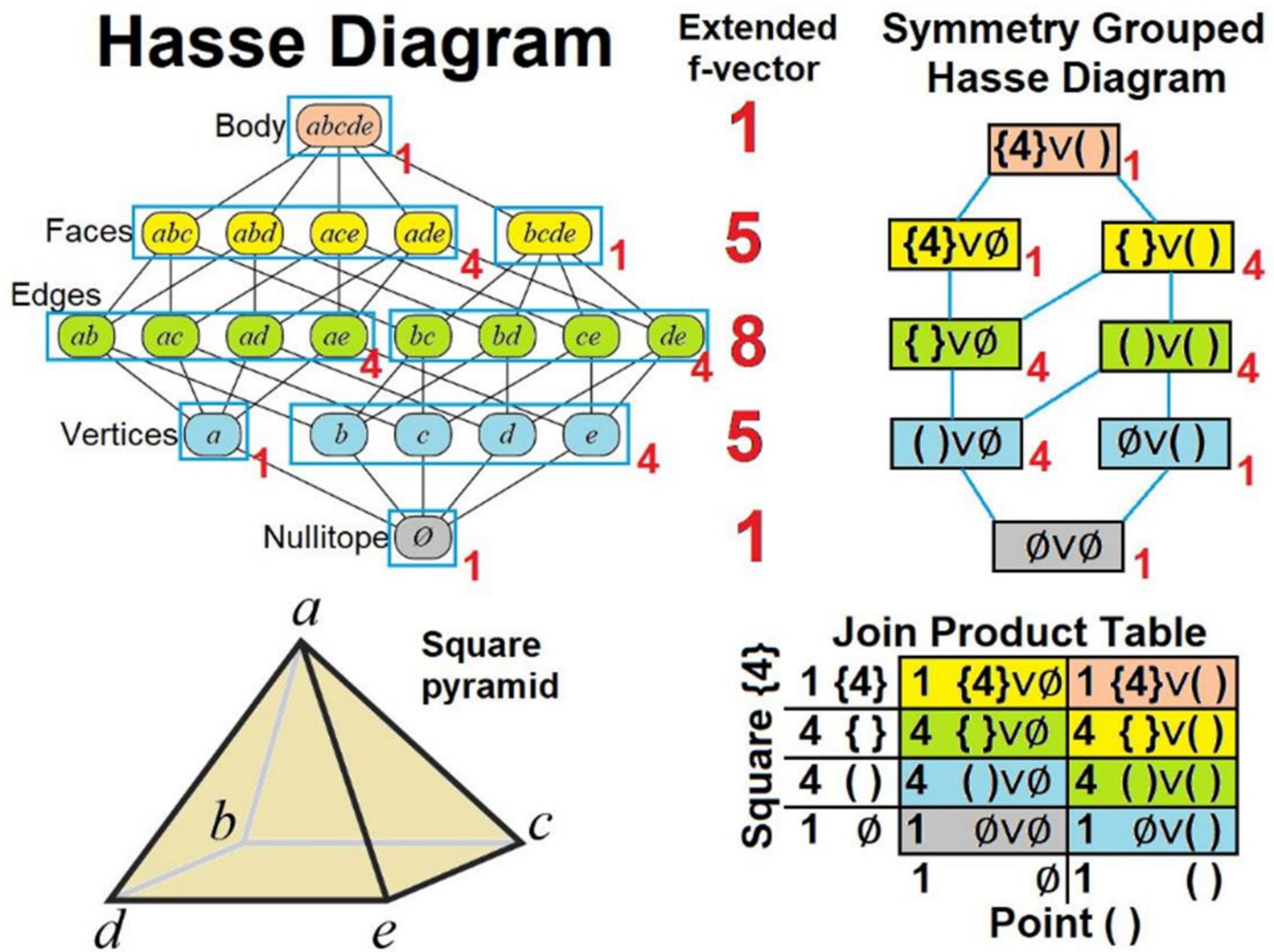
Pyramid: $(1+4x+4x^2+x^3)(1+x) = 1+5x+8x^2+5x^3+x^4 \rightarrow (1,5,8,5,1)$

Prism: $(4+4x+x^2)(2+x) = 8+12x+6x^2+x^3 \rightarrow (8,12,6,1)$

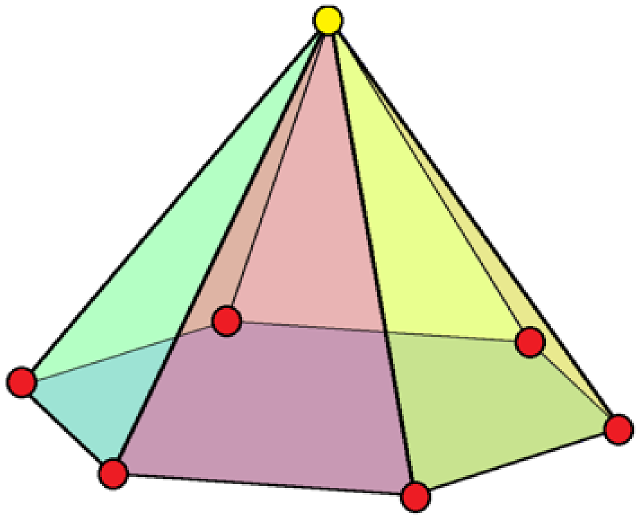




Product tables and Hass diagrams



Hexagonal pyramid, {6} v ( )



{6} v ( )

Characteristic polynomial

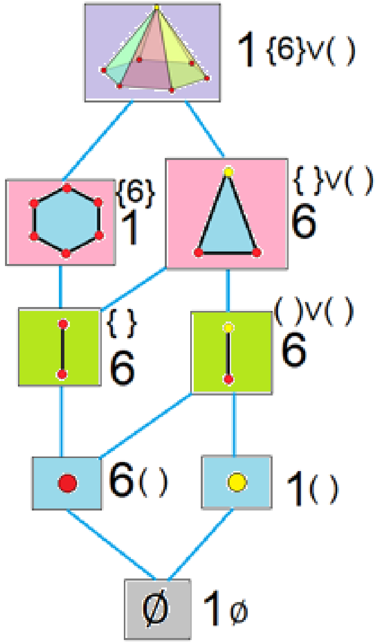
Extended f-vector to polynomial:

base: (1,6,6,1) point: (1,1)

$(1+6x+6x^2+x^3)(1+x) = 1+7x+12x^2+7x^3+x^4$

Join (1,7,12,7,1) 1 nullitope, 7 vertices, 12 edges, 7 faces, 1 body.

Hasse diagram



Join product table

Hexagon		Point: 1 Ø	Point: 1 ( )
{6}	1	1 {6}vØ	1 {6}v( )
{ }	6	6 { }vØ	6 { }v( )
( )	6	6 ( )vØ	6 ( )v( )
Ø	1	1 ØvØ	1 Øv( )

Ø=nullitope , ØvØ = Ø, ØvA = A
( )v( ) = segment,{ }
{ }v( ) = isosceles triangle

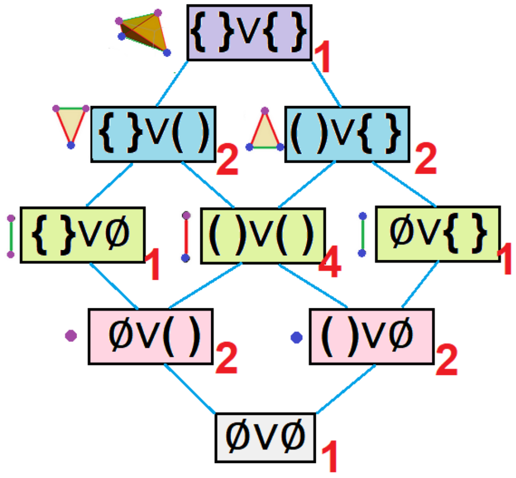
Tetragonal disphenoid – join 2 segments

Fusil similar, but “holes” at segments – creates skew rhombus!

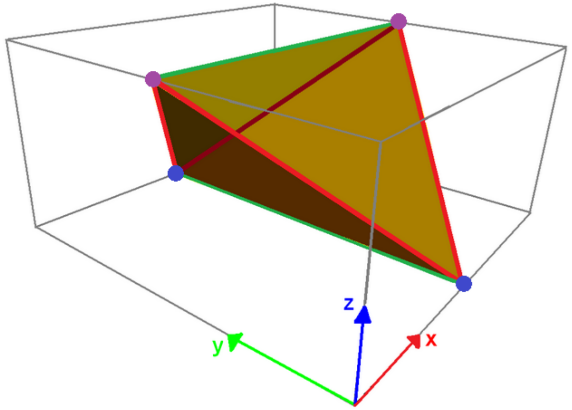
Join product table

1 { }	1 { }vØ	2 { }v( )	1 { }v{ }
2 ( )	2 ( )vØ	4 ( )v( )	2 ( )v{ }
1 Ø	1 ØvØ	2 Øv( )	1 Øv{ }
	1 Ø	2 ( )	1 { }

Hasse Diagram



Tetragonal Disphenoid

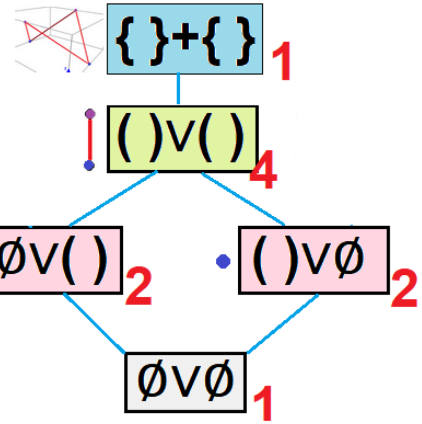


{ }: f-vector (1,2,1)
$(1,2,1)^2 = (1,4,6,4,1)$
4 vertices, 6 edges, and 4 faces

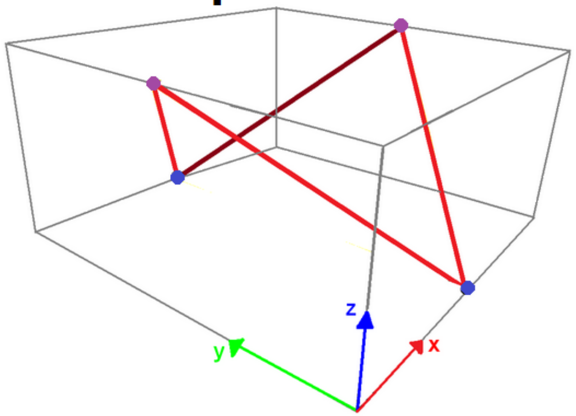
Fusil product table

2 ( )	2 ( )vØ	4 ( )v( )
1 Ø	1 ØvØ	2 Øv( )
	1 Ø	2 ( )

Hasse Diagram



Skew square/rhombus



{ }: f-vector (1,2)
$(1,2)^2 = (1,4,4)$
4 vertices, 4 edges

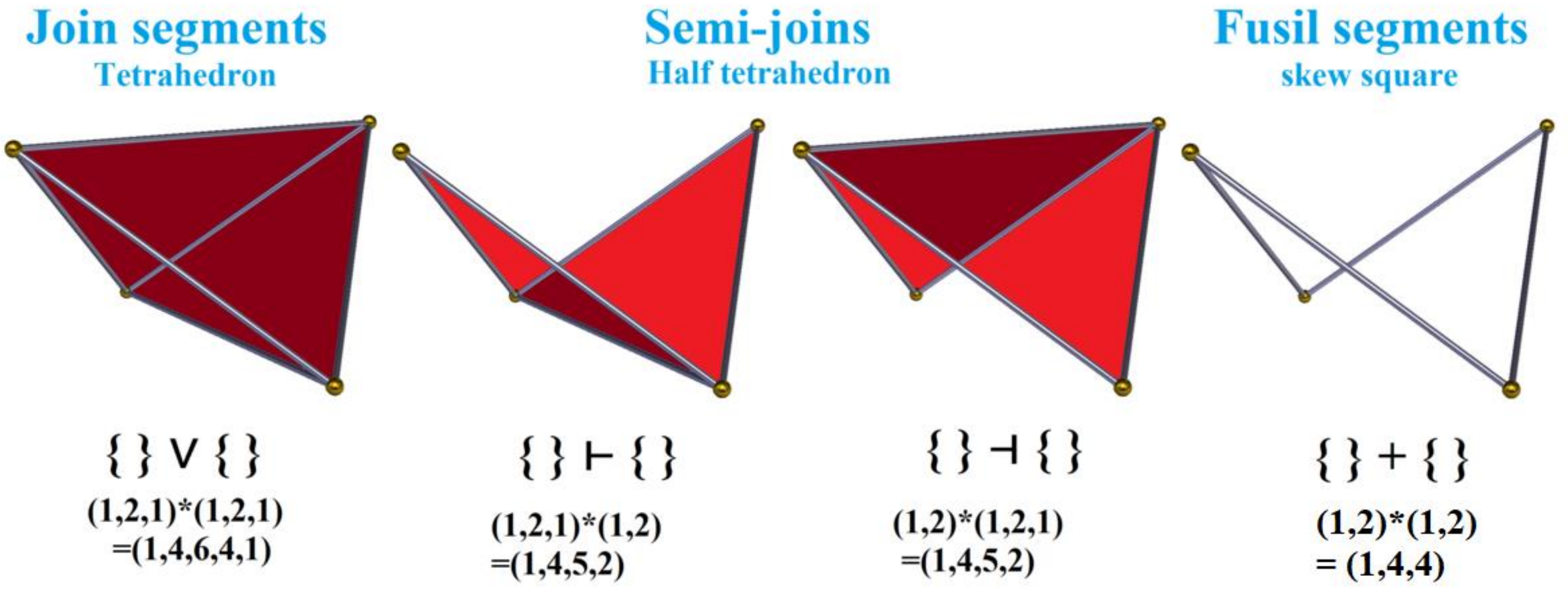
Joins, semi-joins and fusils

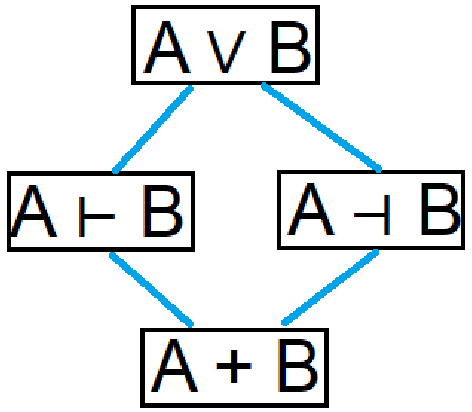
A fusil connects elements of A and B, excluding A and B

Join	$A \vee B$	$(\mathbf{1},f_A,\mathbf{1})*(\mathbf{1},f_B,\mathbf{1})$	
Semi-joins	$A \vdash B$	$(\mathbf{1},f_A,\mathbf{1})*(\mathbf{1},f_B)$	*** Exclude body B
	$A \dashv B$	$(\mathbf{1},f_A)*(\mathbf{1},f_B,\mathbf{1})$	*** Exclude body A
Fusil	$A + B$	$(\mathbf{1},f_A)*(\mathbf{1},f_B)$	

$(A \vdash B) \cup (A \dashv B) = (A \vee B)$ Join is union of two semi-joins

$(A \vdash B) \cap (A \dashv B) = (A + B)$ Fusil is skew intersection





Prism and meet products – pentagon and segment

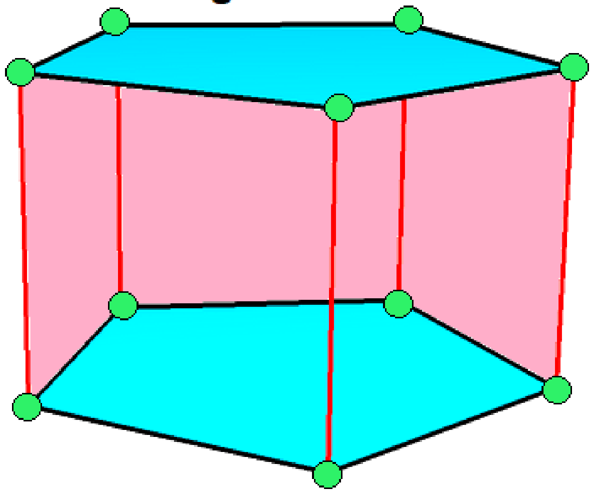
Meet product similar but “holes” at element bodies

{5}∧{ } has 10 vertices, 10 edges, but disconnected?!

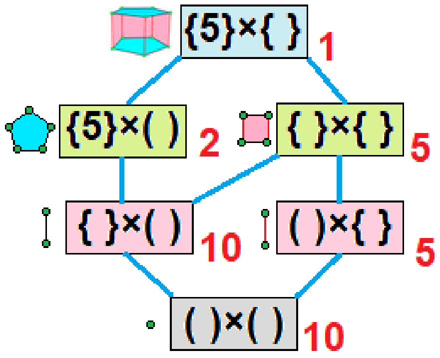
Prism product table

1 {5}	2 {5}×( )	1 {5}×{ }
5 { }	10 { }×( )	5 { }×{ }
5 ( )	10 ( )×( )	5 ( )×{ }
	2 ( )	1 { }

Pentagonal Prism



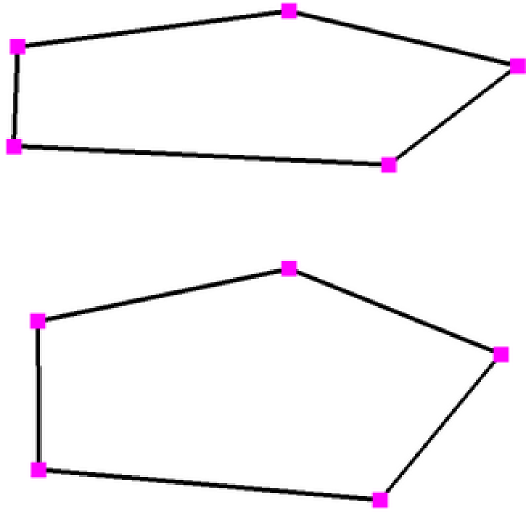
Hasse Diagram



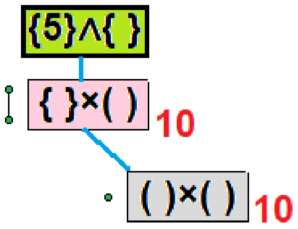
f-vectors:
Pentagon: (5,5,1)
Segment: (2,1)
Prism: (5,5,1)*(2,1)
=(10,10+5,2+5,1)
=(10,15,7,1)

Meet product table

5 { }	10 { }×( )
5 ( )	10 ( )×( )
	2 ( )



Hasse Diagram



f-vectors:
Pentagon: (5,5)
Segment (2)
Meet: (5,5)*(2)
= (10,10)

Prisms, semi-prisms, and meets

A meet is Cartesian product of A and B, excluding A and B

Prism	$A \times B$	$(f_A,\mathbf{1})*(f_B,\mathbf{1})$	
Semi-prisms	$A \leftthreetimes B$	$(f_A,\mathbf{1})*(f_B)$	*** Exclude body B
	$A \rightthreetimes B$	$(f_A)*(f_B,\mathbf{1})$	*** Exclude body A
Meet	$A \wedge B$	$(f_A)*(f_B)$	

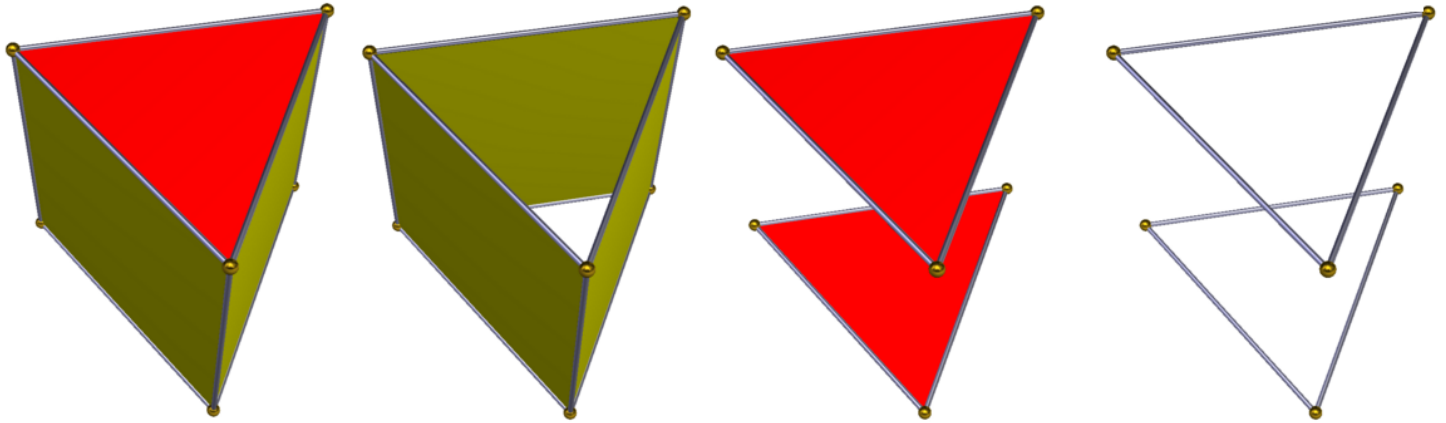
$(A \leftthreetimes B) \cup (A \rightthreetimes B) = (A \times B)$ Prism is union of two semi-prisms

$(A \leftthreetimes B) \cap (A \rightthreetimes B) = (A \wedge B)$ Meet is skew intersection

Triangle-segment prism
Triangular prism

Semi-prisms
Partial triangular prisms

Triangle-segment meet
Skew hexagon

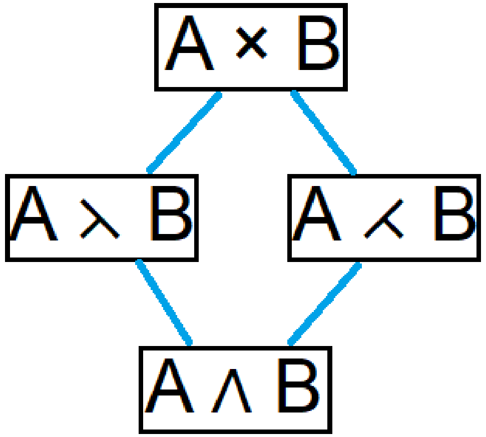


$\{3\} \times \{\,\}$ $(3,3,1)*(2,1) = (6,9,5,1)$

$\{3\} \rightthreetimes \{\,\}$ $(3,3)*(2,1) = (6,9,3)$

$\{3\} \leftthreetimes \{\,\}$ $(3,3,1)*(2) = (6,6,2)$

$\{3\} \wedge \{\,\}$ $(3,3)*(2) = (6,6)$



Summary of polytope operators

f-vector product

Join A∨B
(1,A,1)*(1,B,1)

Identity
∅∨A=A

A⊣B
(1,A)*(1,B,1)

Semijoins

A⊢B
(1,A,1)*(1,B)

Fusil A+B
(1,A)*(1,B)

<---Dual--->

Prism A×B
(A,1)*(B,1)

Identity
( )×A=A

A⋌B
(A)*(B,1)

Semiprisms

A⋋B
(A,1)*(B)

Meet A∧B
(A)*(B)

Dual relations
Join: *(A ∨ B) = (*A) ∨ (*B)
Fusil: *(A + B) = (*A) × (*B)
Prism: *(A × B) = (*A) + (*B)
Meet: *(A ∧ B) = (*A) ∧ (*B)

Union: (A ⊢ B) ∪ (A ⊣ B) = (A ∨ B)
Intersection: (A ⊢ B) ∩ (A ⊣ B) = (A + B)

Union: (A ⋋ B) ∪ (A ⋌ B) = (A × B)
Intersection: (A ⋋ B) ∩ (A ⋌ B) = (A ∧ B)

Names and symbols

Polytope names:

- Join
  - Point pyramid $A \vee (\,)$
  - Segment wedge $A \vee \{\,\}$
  - Polygon+ duo-wedge $A \vee B$
- Fusil
  - Segment fusil $A + \{\,\}$
  - Polygon+ duo-fusil $A + B$
- Prism
  - Segment prism $A \times \{\,\}$
  - Polygon+ duo-prism $A \times B$
- Meet
  - Segment meet $A \wedge \{\,\}$
  - Polygon+ duo-meet $A \wedge B$

Higher product tuples: (Latin prefixes as *n*-tuples)

- Double **duo-** {wedge, fusil, prism, meet}
- Triple **tri-** {wedge, fusil, prism, meet}
- Quadruple **quadri-** {wedge, fusil, prism, meet}
- Quintuple **quinti-** {wedge, fusil, prism, meet}
- Sextuple **sexti-** {wedge, fusil, prism, meet}
- Septuple **septi-** {wedge, fusil, prism, meet}
- Octuple **octi-** {wedge, fusil, prism, meet}
- *n*-tuple ***n*-** {wedge, fusil, prism, meet}

Recursive power notations:

- **Join** $n \cdot A$
- **Fusil** $n\ A$
- **Prism** A^n
- **Meet** $A^{(n)}$

Triple prisms and meets (3-torus)

$\{4\}\times\{4\}\times\{4\} = \{4\}^3$ and $\{4\}\wedge\{4\}\wedge\{4\} = \{4\}^{(3)}$

{4}×{4}×{4} tri-prism product table

1 {4}	16 {4}×( )×{4}	16 {4}×{ }×{4}	4 {4}×{4}×{4}
4 { }	64 { }×( )×{4}	64 { }×{ }×{4}	16 { }×{4}×{4}
4 ( )	64 ( )×( )×{4}	64 ( )×{ }×{4}	16 ( )×{4}×{4}
	4×1 ( )×{4}	4×1 { }×{4}	1×1 {4}×{4}

1 {4}	16 {4}×( )×{ }	16 {4}×{ }×{ }	4 {4}×{4}×{ }
4 { }	64 { }×( )×{ }	64 { }×{ }×{ }	16 { }×{4}×{ }
4 ( )	64 ( )×( )×{ }	64 ( )×{ }×{ }	16 ( )×{4}×{ }
	4×4 ( )×{ }	4×4 { }×{ }	1×4 {4}×{ }

1 {4}	16 {4}×( )×( )	16 {4}×{ }×( )	4 {4}×{4}×( )
4 { }	64 { }×( )×( )	64 { }×{ }×( )	16 { }×{4}×( )
4 ( )	64 ( )×( )×( )	64 ( )×{ }×( )	16 ( )×{4}×( )
	4×4 ( )×( )	4×4 { }×( )	1×4 {4}×( )

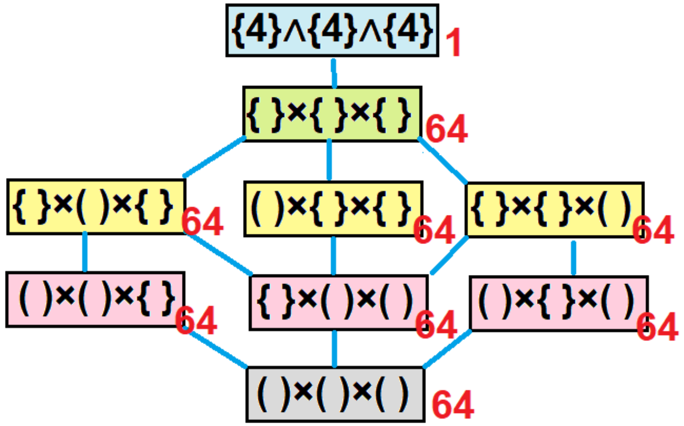
f-vector: $(4,4,1)^3 = (64,192,240,160,60,12,1)$

{4}∧{4}∧{4} tri-meet product table

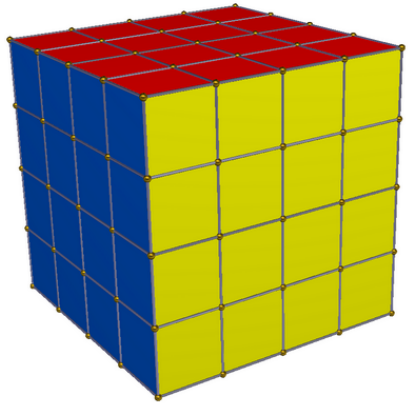
4 { }	64 { }×( )×{ }	64 { }×{ }×{ }
4 ( )	64 ( )×( )×{ }	64 ( )×{ }×{ }
	4×4 ( )×{ }	4×4 { }×{ }

4 { }	64 { }×( )×( )	64 { }×{ }×( )
4 ( )	64 ( )×( )×( )	64 ( )×{ }×( )
	4×4 ( )×( )	4×4 { }×( )

Hasse Diagram



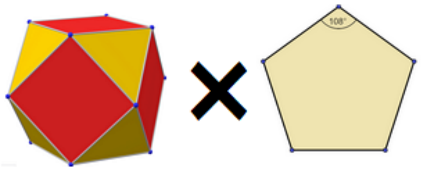
4x4x4 cubic net



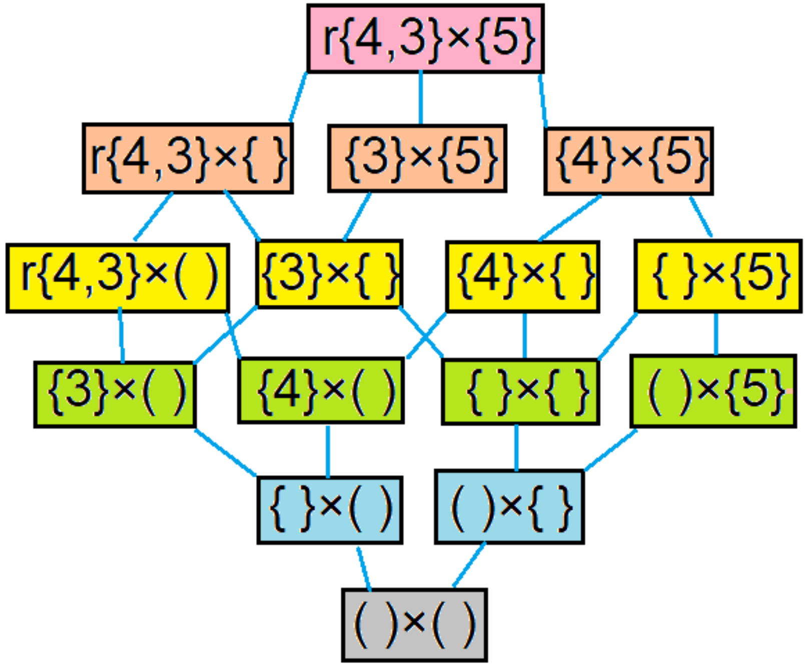
f-vectors:
Square: (4,4)
Tri-meet: $(4,4)^3 = 4^3(1,1)^3$
$= 4^3(1,3,3,1)$
$=(64,192,192,64)$

Cuboctahedron-pentagon prism

5-dimensional prism



Hasse diagram



Prism product table

1 r{4,3}	5 r{4,3}×( )	5 r{4,3}×{ }	1 r{4,3}×{5}
8 {3}	40 {3}×( )	40 {3}×{ }	8 {3}×{5}
6 {4}	30 {4}×( )	30 {4}×{ }	6 {4}×{5}
24 { }	120 { }×( )	120 { }×{ }	24 { }×{5}
12 ( )	60 ( )×( )	60 ( )×{ }	12 ( )×{5}
	5 ( )	5 { }	1 {5}

F-vector product

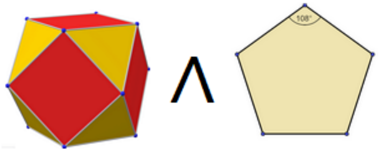
(12,24,6+8,1)*(5,5,1)

=(60, 60+120, 40+30+120+12, 5+40+30+24, 5+8+6,1)

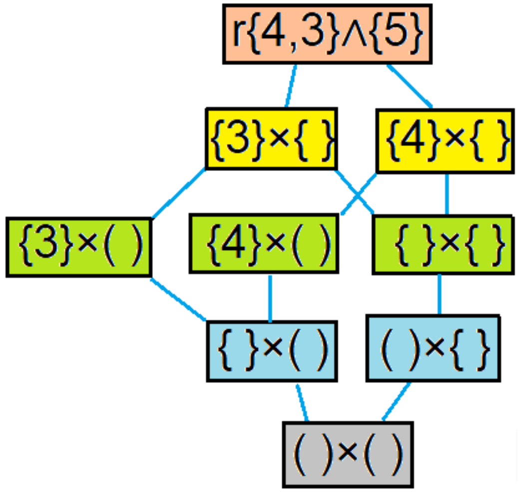
=(60,180,202,99,19,1)

Cuboctahedron-pentagon meet

Skew 4-polytope in 5-dimensions

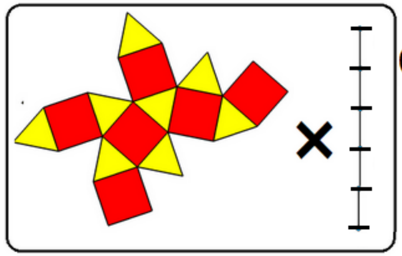


Hasse diagram



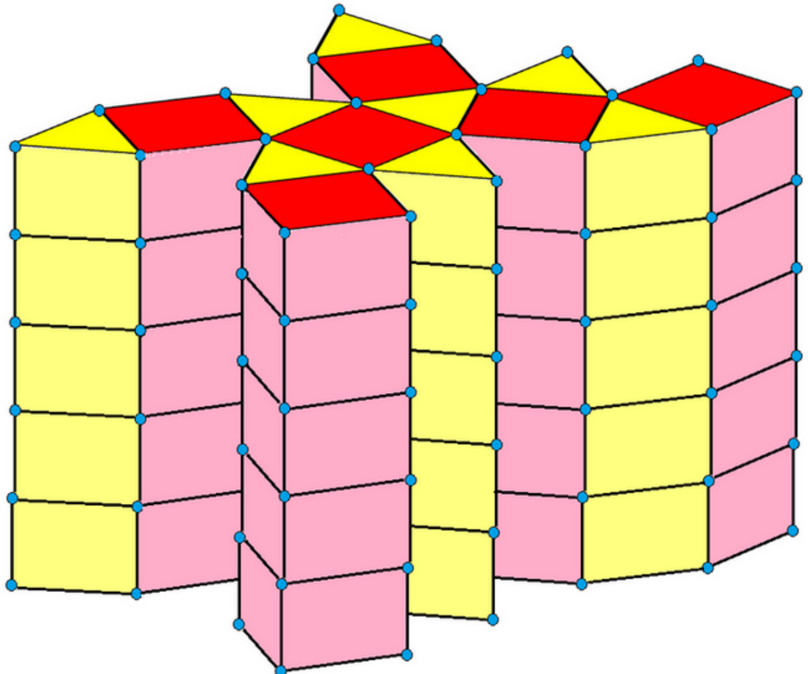
Prism product table

8	{3}	40	{3}×( )	40	{3}×{ }
6	{4}	30	{4}×( )	30	{4}×{ }
24	{ }	120	{ }×( )	120	{ }×{ }
12	( )	60	( )×( )	60	( )×{ }
		5	( )	5	{ }



Net(r{4,3}∧{5})
= Net(r{4,3}) × Net({5})

Cuboctahedron-pentagon meet net



F-vector product
(12,24,6+8)*(5,5)
=(60, 120+60, 40+30+120, 40+30)
=(60,180,190,70)

Conclusion
Polygons, polyhedra, polytopes

A kid's intuitive playground

- Solid geometry you can build!
- Elements you can counts!
- Patterns to discover!

A doorway to:

- Discrete geometry
- Symmetry and Coxeter groups
- Topology
- Combinatorics
- Abstract geometry
- Group theory

Galileo Galilei

[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written.
It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word.

~ Galileo Galilei (1564-1642)