THE COMPLETE ENUMERATION OF FINITE GROUPS OF THE FORM $R_i^2 = (R_i R_j)^{k_{ij}} = 1$

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In this paper, we investigate the abstract group defined by the relations†

$$(1) \qquad R_i^2 = 1 \quad (1 \leqslant i \leqslant m),$$

$$(2) \qquad (R_i R_j)^{k_{ij}} = 1 \quad (1 \leqslant i < j \leqslant m),$$

in order to find what values of the integers k_{ij} will make the group finite.

If some of the relations (2) are absent, we can suppose the corresponding k's to be infinite; but this never happens when the group is finite. If any $k_{ij} = 1$, R_j is merely an alternative name for R_i; therefore we may suppose that $k_{ij} > 1$.

It is convenient to represent the group by a *graph* of dots and links, as in the Table at the end of this paper. The dots represent the generators. The numbers written under certain links are values of k_{ij}. Whenever a link is not so numbered, we understand that $k_{ij} = 3$. Whenever two dots are not (directly) linked, we understand that $k_{ij} = 2$.

The group is said to be *irreducible* or *reducible* according as its graph is connected or disconnected. If reducible, it is the direct product of two or more irreducible groups, represented by the connected pieces of the graph.

With the help of certain lemmas, we shall prove the following

Theorem‡. *The only irreducible finite groups of the form*

$$R_i^2 = (R_i R_j)^{k_{ij}} = 1$$

are

$$[3^n], \quad [3^n, 4], \quad [k], \quad [3, 5], \quad [3, 4, 3], \quad [3, 3, 5],$$

$$\begin{bmatrix} 3^n \\ 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}.$$

(The numbers occurring in these symbols are the values of those k's that are greater than 2. For details, see the Table.)

* Received 30 October, 1933; read 16 November, 1933.

† In compact form, $(R_i R_j)^{k_{ij}} = 1$ $(1 \leqslant i \leqslant j \leqslant m,\ k_{ii} = 1)$.

‡ *Cf.* H. S. M. Coxeter, *Annals of Math.*, 35 (1934), 601 (Theorem 9). We shall refer to this paper as D.g.g.r.

LEMMA 1. *If the particular form*

$$\sum_1^n b_j x_j^2,$$

where $$b_1 = b_2 = \dots = b_p \neq b_i$$
$$(p \geqslant 1, \quad i = p+1, p+2, \dots, n),$$

is invariant under a certain orthogonal substitution, then $\sum_1^p x_j^2$ *is likewise invariant.*

Since the substitution is orthogonal, $\sum_1^n x_j^2$ must be invariant, and so also must the difference $\sum_{p+1}^n (b_j - b_1) x_j^2$.

The equation $$\sum_{p+1}^n (b_j - b_1) x_j^2 = 0$$

represents a (p times degenerate) *cone*, whose "vertex" is the p-space

$$x_{p+1} = x_{p+2} = \dots = x_n = 0.$$

Since the cone is invariant, this p-space must also be invariant. Therefore the form $\sum_1^n x_j^2$ will still be invariant when we replace all save the first p x's by zero.

LEMMA 2. *Any linear transformation that leaves invariant the two particular forms*

(1) $$\sum_1^n \epsilon_j y_j^2 \quad (\epsilon_j^2 = 1),$$

(2) $$\sum_1^n c_j y_j^2 \quad (c_j > 0 \textit{ for all } j),$$

also leaves invariant $\sum_1^n y_j^2$.

If $b_1, b_2, \dots, b_n$ are real non-vanishing numbers, such that the form $\sum_1^n b_j x_j^2$ is invariant under a certain orthogonal substitution, then Lemma 1 shows that the sums of the positive and negative terms are separately invariant. Therefore

$$\sum_1^n |b_j| x_j^2$$

is invariant. Since an orthogonal substitution is merely a linear transformation that leaves $\sum_1^n x_j^2$ invariant, Lemma 2 can now be deduced by putting

$$|b_j| = c_j^{-1}$$

$$b_j = \epsilon_j c_j^{-1},$$

$$x_j = c_j^{\frac{1}{2}} y_j.$$

Abbreviation. For "group generated by reflections" we write g.g.r.

LEMMA 3. *Every finite g.g.r. in the generalized Minkowskian space* $S^s T^t$ is simply isomorphic with a g.g.r. in the Euclidean space S^{s+t}.*

Let the general point of $S^s T^t$ be $(x_1, x_2, \ldots, x_{s+t})$, at distance

$$\sum_1^{s+t} \epsilon_j x_j^2 \quad (\epsilon_j^2 = 1,\ \Sigma \epsilon_j = s - t)$$

from the origin.

It is well known that any finite linear group leaves invariant a positive definite quadratic form, *e.g.* the sum of all transforms of Σx_j^2. This must hold, in particular, for a finite group of congruent transformations in $S^s T^t$. By a suitable change of coordinates, under which the general point becomes $(y_1, y_2, \ldots, y_{s+t})$, at distance

$$\sum_1^{s+t} \epsilon_j y_j^2$$

from the (new) origin, the invariant form becomes (say)

$$\sum_1^{s+t} c_j y_j^2 \quad (c_j > 0).$$

By Lemma 2, we can now assert the invariance of $\sum_1^{s+t} y_j^2$.

We thus have a group of congruent transformations in $S^s T^t$, leaving $\sum_1^{s+t} y_j^2$ invariant. By giving the variables y_j a new geometrical interpretation, we can regard the same algebraic substitutions as congruent transformations in S^{s+t}, leaving $\sum_1^{s+t} \epsilon_j y_j^2$ invariant.

A reflection is characterized by the fact that it leaves invariant every point whose coordinates satisfy a certain linear equation. Therefore, reflections remain reflections when we pass from $S^s T^t$ to S^{s+t}.

* H. S. M. Coxeter and J. A. Todd, *Proc. Camb. Phil. Soc.*, 30 (1934), 1–3. The reflection in the prime $\Sigma a_j x_j = 0$ is the transformation

$$x_i' = x_i - 2\epsilon_i a_i \lambda \quad (i = 1, 2, \ldots, s+t),$$

where $\lambda = \Sigma a_j x_j / \Sigma \epsilon_j a_j^2$.

Lemma 4. *Every finite group of the form $R_i^2 = (R_i R_j)^{k_{ij}} = 1$ can be generated by reflections in the bounding primes of a spherical simplex.*

We know* that there exist, in some generalized Minkowskian space $S^s T^t$ $(s+t \leqslant m)$, $m+1$ points $A^0, A^1, \ldots, A^m$, such that

$$A^0 A^i = 1 \qquad (1 \leqslant i \leqslant m),$$

$$A^i A^j = 2 \cos(\pi/2k_{ij}) \qquad (1 \leqslant i < j \leqslant m).$$

Let a^i $(1 \leqslant i \leqslant m)$ denote the prime, through A^i, perpendicular to $A^0 A^i$. Then it is easily seen that a^i, a^j are inclined at an angle π/k_{ij}. The primes a^i cut off a certain region around A^0. (If closed, this region is a polytope, and A^0 is its in-centre.) Reflection in any prime a^i gives a new region, congruent† to the first. Reflection in any other bounding prime of the new region gives a third region; and so on. Since all the dihedral angles are of the form π/k, the regions will fit together without overlapping, and there will be no interstices‡. In fact, the region bounded by the primes a^i is a *fundamental region* for the group generated by the reflections in these primes.

Let R_i denote the reflection in the prime a^i. The relations (1) and (2) evidently hold. The R's, so defined, may perhaps satisfy other relations, not deducible from these§. But we can assert that the g.g.r. is simply isomorphic either with the abstract group defined by (1) and (2) or with a factor group thereof. Hence, if the abstract group is finite, the g.g.r. is *a fortiori* finite.

By Lemma 3, the g.g.r. occurs in Euclidean space. By Theorem 8 of *Discrete groups generated by reflections*‖, it is simply isomorphic with the whole abstract group. Since the origin is invariant¶, the group can be regarded as operating in spherical space. By Lemma 4.7 of the paper just cited**, if the number of dimensions is taken as small as possible, the spherical fundamental region is a simplex.

Our theorem now follows from the enumeration of *Groups whose fundamental regions are simplexes*††.

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* Coxeter and Todd, *loc. cit.*, 1.

† Or, rather, *enantiomorphous*.

‡ *Cf.* D.g.g.r., 596.

§ The argument used in D.g.g.r. shows that such extra relations will appear only if the part of space filled by the fundamental region and its transforms is multiply-connected.

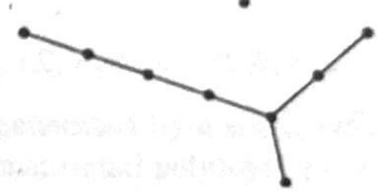
‖ D.g.g.r., 599.

¶ In Euclidean (or Minkowskian) space, every finite group of congruent transformations leaves invariant the centroid of all the transforms of a point of general position.

** D.g.g.r., 597.

†† *Journal London Math. Soc.*, 6 (1931), 132–134. For a fuller account, see *Proc. London Math. Soc.* (2), 34 (1932), 144–151. In both these papers (the former, the penultimate line on p. 132; the latter, the last line on p. 136), a_{ii} should be a_{ij}.

TABLE OF IRREDUCIBLE FINITE GROUPS OF THE FORM $R_i^2 = (R_iR_j)^{k_{ij}} = 1$.

Symbol		Graph	Order*
$[3^{m-1}]$†	$(m \geqslant 1)$		$(m+1)!$
$[3^{m-2}, 4]$	$(m \geqslant 2)$		$2^m m!$
$[k]$‡	$(k \geqslant 5)$		$2k$
$[3, 5]$§			120
$[3, 4, 3]$			1152
$[3, 3, 5]$			14400
$\begin{bmatrix} 3^{m-3} \\ 3 \\ 3 \end{bmatrix}$‖	$(m \geqslant 4)$		$2^{m-1} m!$
$\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$¶			51840
$\begin{bmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$			2903040
$\begin{bmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$			696729600

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* *Proc. London Math. Soc.* (2), 34 (1932), 160, 159. All the orders save the last three can be deduced from the theory of regular polytopes. For the rest, see *Phil. Trans. Royal Soc.* (A), 229 (1930), 381–384.

† The symmetric group of degree $m+1$. [ ] is the group of order 2; its graph is a single dot. m is the number of dots in the graph, *i.e.* the number of generators.

‡ The dihedral group of order $2k$. [3] and [4] have already occurred above.

§ The extended icosahedral group.

‖ $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ is the same as [3, 3].

¶ The group of the twenty-seven lines on the cubic surface.