4.7. Unconventional configurations

In this section we shall consider several families of objects that we shall call "configurations" even though they do not fit the definition of that word accepted in all the other sections of this book.

The first of these families are "configurations of points and circles". Some examples are shown in Figures 4.7.1 and 4.7.2. In analogy to configurations of points and lines we may denote them by a symbol such as (p_q, n_k), where p, n are the numbers of points and of circles and q, k are the number of circles incident with each point and the number of points incident with each circle; in case the numbers are equal, we use the notation (n_k). Hence the three configurations shown are (4_3), $(8_3, 6_4)$, and (10_4).

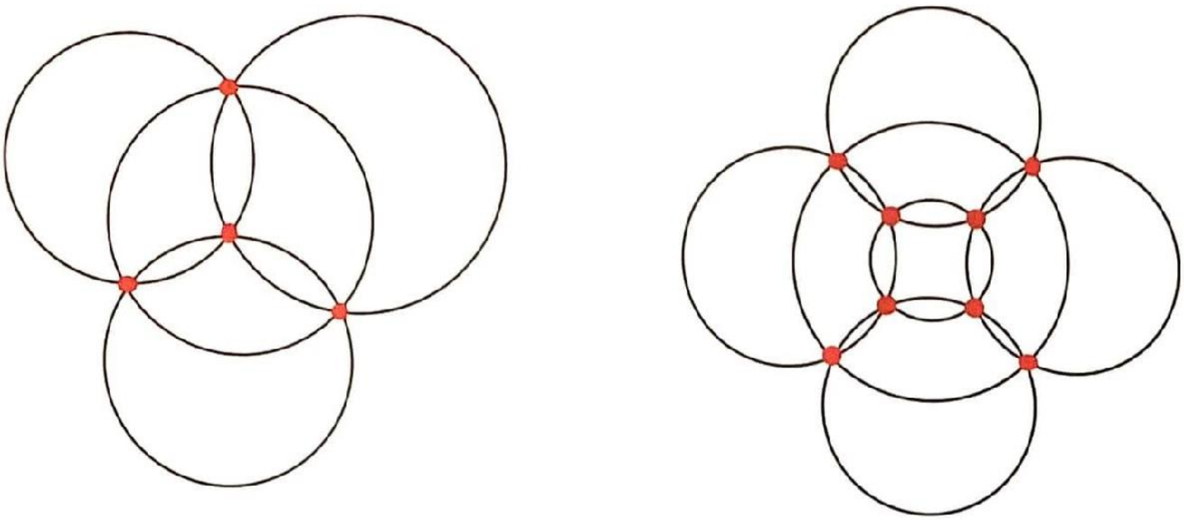
Several aspects of configurations of points and circles deserve notice.

First, such configurations are generalizations of configurations of points and lines in a very direct way: Every configuration of points and lines in the projective plane can be shown as a configuration of antipodal pairs of points and great circles in the model of the projective plane on the sphere; a stereographic projection then maps this into a configuration of points and circles in the plane. However, these are only very special cases of such configurations—none of those in Figures 4.7.1 and 4.7.2 are of this kind.

Second, in all but name, configurations of points and circles made their appearance before configurations of points and lines. For example, the configuration in Figure 4.7.1(b) is an illustration of a theorem of A. Miquel [**171**] dating to 1844, asserting that if four pairwise intersections of four circles are concyclic, the other four intersections of the same pairs are concyclic as well. This is one of several results of Miquel, some of which have been greatly generalized by many writers, starting with Clifford [**42**] in 1871 and de Longchamp [**147**] in 1877. One of the achievements is the so-called "chains of theorems" bearing the names of Clifford and de Longchamps. The former establishes the existence of configurations of points and circles $((2^{n-1})_n)$ for all $n \geq 1$. The cases $n = 1$ or 2 are trivial, and $n = 3$ is shown in Figure 4.7.1(a). For more recent works on this and related topics see, for example, Ziegenbein [**239**], Rigby [**190**], Longuet-Higgins [**148**], Longuet-Higgins and Parry [**149**], and references given therein to other works.

Third and last—why is there no greater activity regarding these configurations? We venture to guess that the preoccupation with just a few specific results (such as the "chains of theorems") tended to discourage more general inquiries. There are various subclasses of circle configurations that may well be worth investigating: Are pairs of circles required to intersect twice, are touching circles allowed, can disjoint circles appear, are straight lines admitted, does one wish to consider symmetries in the inversive plane—the choices and possibilities are very wide and almost entirely unexplored. (The inversive plane seems to be an appropriate setting for many of the considerations of symmetries of configurations of points and circles; see, for example, Coxeter [**47**], Eves [**67**], Yaglom [**230**].)

The configuration (10_4) in Figure 4.7.2 is an example of configurations $((2n)_{n-1})$ that exist for all $n \geq 5$ and exhibit remarkable symmetry in the inversive plane. The (10_4) configuration has a single orbit of points and a single orbit of circles under inversive transformations. The author does not know what other configurations are as symmetric, but probably there are many additional ones.



(a)

(b)

Figure 4.7.1. Configurations of points and circles. (a) An (4_3) configuration. (b) An $(8_3, 6_4)$ configuration.

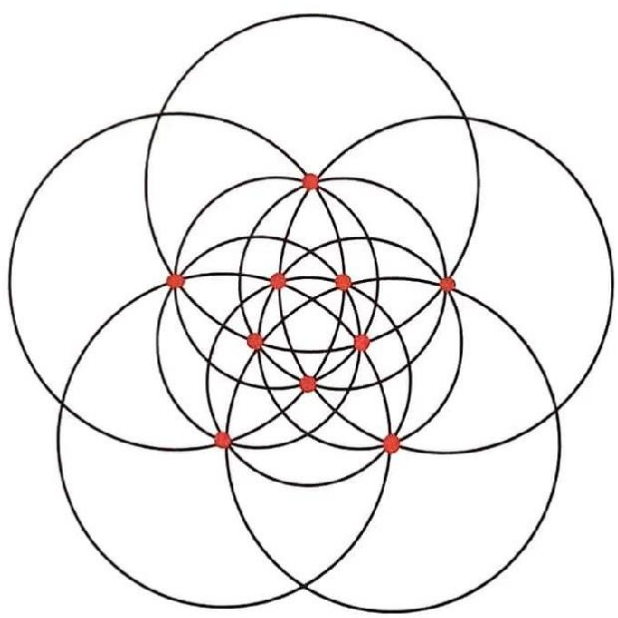


Figure 4.7.2. A (10_4) configuration of points and circles.