The Evolution of Coxeter-Dynkin Diagrams

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Dedicated to Jacques Tits on the occasion of his 60th birthday

1. Introduction

The use of trees as diagrams for groups was anticipated in 1904, when C. Rodenberg was commenting on a set of models of cubic surfaces. He was analyzing the various rational double points that can occur on such a surface. In 1931, I used these diagrams in my enumeration of kaleidoscopes, where the dots represent mirrors. E.B. Dynkin re-invented the diagrams in 1946 for the classification of simple Lie algebras. For instance, the graphs

are related to Wilhelm Killing's algebras

$$A_4, \quad D_5, \quad E_6, \quad E_7, \quad E_8.$$

Many other applications have arisen since that time (see, for instance, Hazewinkel, Hesselink, Siersma and Veldkamp 1977).

Since Rodenberg made use of double-sixes, it seems appropriate to begin this history with a summary of the pioneering work of Arthur Cayley, George Salmon, Jacob Steiner, and Ludwig Schläfli.

2. The non-singular cubic surface

Since a straight line usually intersects a cubic surface in three points, a line that contains four points of the surface must lie entirely on the surface. In 1849, such considerations persuaded Cayley that, since the whole projective 3-space contains ∞^4 lines, the general cubic surface should contain a finite number of them. He communicated this idea to Salmon, who replied that this finite number is 27. Five years later there was an equally fruitful correspondence in Switzerland between STEINER (1857) and SCHLÄFLI (1858). Steiner showed that a certain set of 9 among the 27 lines can be regarded as the intersections of two trihedra, that is, two sets of three planes. Since each of the 27 lines lies in 5 such planes, there are altogether 45 of them. Moreover, there are 120 of his pairs of 'conjugate' trihedra.

In his reply, Schläfli described his famous *double-six*

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix} \tag{2.1}$$

such that two of these twelve lines intersect if and only if their symbols occur neither in the same row nor in the same column; thus a_1 intersects b_2 but is *skew* to a_2 and b_1. The planes a_1b_2 and a_2b_1 intersect in a line $c_{12}(=c_{21})$ which, containing four points of the surface, lies entirely on it. By considering the possible intersections of other lines with the plane a_1b_2, Schläfli easily verified Salmon's number 27: the 2×6 lines a_ν, b_ν, and the $\binom{6}{2}$ lines $c_{\mu\nu}(\mu<\nu)$.

Any two of the 27 lines are either intersecting or skew. The intersecting pairs are

$$a_1b_2,\ a_1c_{12},\ b_1c_{12},\ c_{12}c_{34},\ \text{etc.}$$

while the skew pairs are

$$a_1b_1,\ a_1a_2,\ b_1b_2,\ a_1c_{23},\ b_1c_{23},\ c_{12}c_{13},\ \text{etc.}$$

(Schläfli 1958, p. 213). In particular, two lines of the form $c_{\mu\nu}$ intersect if their subscripts have no common digit, but are skew if they have one common digit.

3. THE WEYL GROUP E_6 OF ORDER 51840

The group of automorphisms of this configuration of 27 lines was investigated by JORDAN (1870, p. 317), MASCHKE (1888, p. 320), BURKHARDT (1891, p. 317) and DICKSON (1916, p. 348). Although these men made successively simpler choices of generators and relations, it seems that none of them saw how to generate this group by six involutions, suitably chosen from among the 36 involutions observed by BURKHARDT (1891, pp. 324-326). Each of these 36 involutions interchanges the two rows of a double-six while leaving invariant the remaining fifteen lines. Those authors apparently failed to notice the simple structure of the set of 36 double-sixes, namely, that any two of them contain either 4 or 6 common lines: 4 forming a 'double-two' (two skew pairs of intersecting lines) or 6 forming a 'grid' (two intersecting triads of skew lines). When there are 4 common lines, the corresponding involutions *commute* ($AB = BA$, or $A \underset{2}{\leftrightarrow} B$); but when there are 6, the involutions are *braided* ($ABA = BAB$, or $B^A = A^B$, or $A \underset{3}{\leftrightarrow} B$). For instance, the double-six (2.1) shares with

$$\begin{bmatrix} a_1 & b_1 & c_{23} & c_{24} & c_{25} & c_{26} \\ a_2 & b_2 & c_{13} & c_{14} & c_{15} & c_{16} \end{bmatrix}$$

the 4 lines a_1, a_2, b_1, b_2, and with

$$\begin{bmatrix} a_1 & a_2 & a_3 & c_{56} & c_{46} & c_{45} \\ c_{23} & c_{13} & c_{12} & b_4 & b_5 & b_6 \end{bmatrix}$$

the 6 lines $a_1, a_2, a_3, b_4, b_5, b_6$. The corresponding involutions are the permutations

$$N = (a_1 b_1)(a_2 b_2)(a_3 b_3)(a_4 b_4)(a_5 b_5)(a_6 b_6),$$
$$N_{12} = (a_1\, a_2)(b_1\, b_2)(c_{13}\, c_{23})(c_{14}\, c_{24})(c_{15}\, c_{25})(c_{16}\, c_{26}),$$
$$N_{123} = (a_1\, c_{23})(a_2 c_{13})(a_3 c_{12})(b_4\, c_{56})(b_5\, c_{46})(b_6\, c_{45})$$

(Gruber and Wills, 1983, p. 112). Cartan (1894, p. 331) called them T, S_{12}, S_{123}. One might say briefly

$$N = (a\, b), \quad N_{12} = (1\,2),$$

and observe that N transforms N_{123} into N_{456}. Three involutions A, B, C may conveniently be said to form a *braided triad* if

$$ABA = BAB = C;$$

for then also $BCB = CBC = A$ and $CAC = ACA = B$. Among the 36 involutions arising from the double-sixes, typical commutative pairs are

$$NN_{12},\ N_{12}N_{123},\ N_{12}N_{34},\ N_{12}N_{345},\ N_{123}N_{145},$$

and typical braided triads are

$$N_{12}N_{13}N_{23},\ NN_{123}N_{456},\ N_{12}N_{134}N_{234}.$$

In particular, two involutions of the form $N_{\lambda\mu\nu}$ are commutative or braided according as their subscripts have an odd or even number of common digits.

4. Singularities

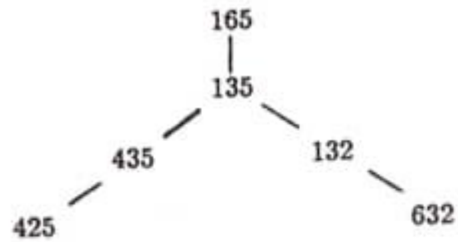
Rodenberg (1904, pp. 5, 32) used the notation $\lambda\mu\nu$ for the double-six corresponding to $N_{\lambda\mu\nu}$, and invented a diagram in which symbols for two double-sixes are linked when they share 6 lines (so that the involutions are braided) but not linked when they share only 4 lines (so that the involutions commute). He was investigating the possible singularities of cubic surfaces. In this project he essentially anticipated the discovery by Du Val (1934) of a connection between double points

$$B_3,\ B_4,\ B_5,\ B_6,\ U_6,\ U_7,\ U_8,\ U_9$$

(*Biplanar* or *Uniplanar*) and reflection groups

$$A_2,\ A_3,\ A_4,\ A_5,\ D_4,\ D_5,\ E_6,\ \tilde{E}_6$$

(see also ARNOLD 1974, pp. 21, 24; FISCHER 1986b, p. 13). For instance, Rodenberg described the uniplanar double points U_8 by a diagram



and mentioned that U_9 could be derived by appending 164, just above 165, so as to make a 'triquetra' (COXETER 1988, p. 23, 26) which would be symmetrical by the permutation

$$(1\ 3\ 5)\ (2\ 4\ 6).$$

His seemingly unnatural ordering of digits (165 rather than 156, and so on) was determined by his insistence that, when two neighbouring triads have two common digits, these should occur in the same two positions.

In a trigonally symmetrical model of a cubic surface with 27 real lines (FISCHER 1986a, Figures 10, 11, 12) those seven double-sixes appear as seven 'holes' or 'passages' (Durchgänge), any two of which are visibly adjacent or non-adjacent according as the corresponding double-sixes share 6 or 4 lines.

Rodenberg's diagram for U_8 exhibits six involutions which suffice to generate the Weyl group E_6. He happened to choose six which are all of the form $N_{\lambda\mu\nu}$. But instead of

$$\begin{array}{ccccccccc} & & & & N_{156} & & & & \\ & & & & | & & & & \\ N_{245} & \text{———} & N_{345} & \text{———} & N_{135} & \text{———} & N_{123} & \text{———} & N_{236} \end{array}$$

he could just as well have chosen

$$\begin{array}{ccccccccc} & & & & N_{123} & & & & \\ & & & & | & & & & \\ N_{12} & \text{———} & N_{23} & \text{———} & N_{34} & \text{———} & N_{45} & \text{———} & N_{56} \end{array}$$

so as to reveal Schläfli's symmetric subgroup of degree 6, generated by the transpositions

$$(1\,2),\quad (2\,3),\quad (3\,4),\quad (4\,5),\quad (5\,6).$$

The same diagram, with the six generators differently named, appeared independently in one of my early works (COXETER 1932, p. 164). While that paper was in press, I realized that, instead of naming the generators, we could simply use dots. In fact, the graph

symbolizes a presentation for the group $[3^{2,2,1}] \cong E_6$ (COXETER 1931, p. 133; 1934, p. 618; 1935, p. 25): the dots represent involutory generators, and two of them are linked or not linked according as the corresponding involutions are braided or commutative. By removing generators successively from the right, we can similarly present subgroups D_5, A_4 and $A_2 \times A_1$ (RODENBERG 1904, pp. 31, 27).

5. THE DEL PEZZO SURFACES

P. DEL PEZZO (1887, pp. 243-253) regarded the cubic surface in 3-space as the member F_2^3 of a family surfaces F_2^n of order n in n-space for $n \leqslant 9$. This surprising inequality arises from del Pezzo's ingenious argument, developed synthetically (without any direct appeal to coordinates). He first observed that an algebraic surface, being a 2-manifold, must be either a *cone* or another kind of *ruled* surface or a *non-ruled* surface, and that these three categories are preserved by projection from a point into a hyperplane. The *order* of the surface (its maximal number of intersections with a general ~~line~~) also is preserved, (n−2)-flat except when the centre of projection lies on the original surface, in which case the order is diminished by 1. In particular, a surface F_2^n is projected, from a point on itself, onto a surface F_2^{n-1} in $(n-1)$-space. Projecting this again, and continuing, we eventually reach a cubic surface F_2^3 in 3-space. When F_2^n is projected onto F_2^{n-1}, a set of m mutually skew lines on the former yields a set of $m+1$ such lines on the latter, the extra one arising from the neighbourhood of the centre of projection, or from the tangent plane there. Since F_2^3 has a maximal set of *six* skew lines (such as $a_1, \cdots, a_6$), F_2^4 has a maximal set of five, F_2^5 of four, F_2^6 of three, F_2^7 of two, F_2^8 contains at most one line, and F_2^9 has none. Finally, a surface of order 10 in 10-space must be ruled (or a cone) because a non-ruled F_2^{10} would yield a line on F_2^9. This shows that non-ruled surfaces F_2^n occur only when $n \leqslant 9$.

Since the tangent plane to F_2^n at the centre of projection yields a line on F_2^{n-1}, all the lines on F_2^n have the same relations of incidence as those lines on F_2^{n-1} which are skew to one. Among the 27 lines on F_2^3, each is skew to 16 others; for instance, b_6 is skew to

a_6

b_1 b_2 b_3 b_4 b_5

c_{12} c_{23} c_{34} c_{45}

c_{13} c_{24} c_{35}

c_{14} c_{25}

c_{15}

and we can use these same symbols for the 16 lines on the quartic surface F_2^4 (which is the intersection of two quadric 3-folds). Similarly, the quintic surface F_2^5 contains 10 lines having the same incidence as those lines on F_2^4 which are skew to one, say a_6, namely

$$c_{\mu\nu} \quad (\mu < \nu < 6),$$

these being the 10 in the last four lines of the above list. The sextic surface F_2^6 contains 6 lines having the same incidence as those on F_2^5 which are skew to (say) c_{45}, namely the 'double-three'

$$\begin{matrix} c_{14} & c_{24} & c_{34} \\ c_{15} & c_{25} & c_{35} \end{matrix}$$

which may be regarded as a skew hexagon $c_{14}\,c_{35}\,c_{24}\,c_{15}\,c_{34}\,c_{25}$. The septimic surface F_2^7 contains 3 lines having the same incidences as those on F_2^6 which are skew to (say) c_{25}, namely

$$\begin{matrix} & c_{24} & \\ c_{15} & & c_{35} \end{matrix}$$

Here a new situation arises because c_{24} intersects both c_{15} and c_{35}, which are skew. Accordingly, there are two distinct octavic surfaces F_2^8: one containing just one line, c_{15} or c_{35}, and the other none. Finally, the unique novenic surface F_2^9 again contains no line.

6. Uniform polytopes

A few years before his work on the cubic surface, Schläfli (1852, pp. 215, 224) discovered the regular polytopes, including the *simplex* α_n, the *cross polytope* β_n and the *n-cube* γ_n. Gosset (1900) enumerated the uniform (or semi-regular) polytopes, later called p_{21} ($p \leqslant 5$, in $p+4$ dimensions), each of which is the vertex figure of the next (Coxeter 1932, pp. 130, 163). Their facets consist of regular simplexes $p_{20} = \alpha_{p+3}$ and cross polytopes $p_{11} = \beta_{p+3}$; their numbers of vertices,

for	$p =$	$-2,$	$-1,$	$0,$	$1,$	$2,$	$3,$	$4,$	$5,$
are		$3,$	$6,$	$10,$	$16,$	$27,$	$56,$	$240,$	$\infty.$

Unaware of Gosset's discovery, I rediscovered these 'pure Archimedean' polytopes about 1925. Three years later I gave a talk on them at one of Professor H.F. Baker's Saturday afternoon tea-parties (in the Arts School of the University of Cambridge), attended by about a dozen students and colleagues. When I had written the above sequence of numbers on the blackboard, a colleague who was well acquainted with algebraic geometry (J.G. Semple?) excitedly remarked that the first six are precisely the numbers of lines on the del Pezzo surfaces F_2^{5-p}.

If, instead of British geometers, my audience had consisted of French analysts, someone would just as eagerly remarked that the same numbers appeared in the work of Elie Cartan (1894, p. 343) as factors of the order

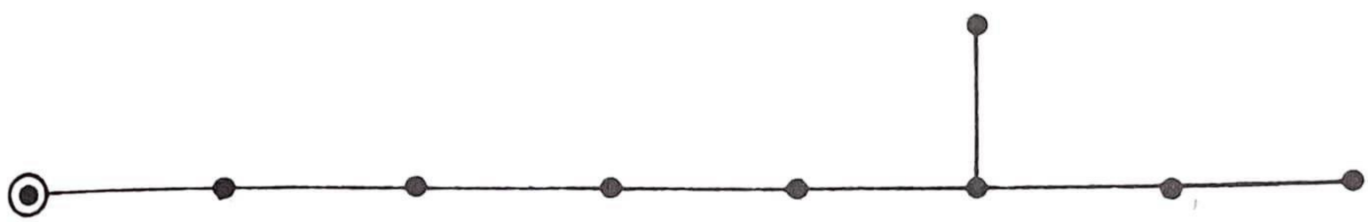
$$240 \times 56 \times 27 \times 16 \times 10 \times 6 \times 2$$

of the Weyl group E_8, which was already implicit in 'The greatest mathematical paper of all time' (Coleman 1989, Killing 1889, pp. 23, 28-29; Weyl 1935, Appendix).

The symbol p_{qr}, of which p_{21} is a special case, first arose as a name for the polytope with triangular faces whose vertex figure is $(p-1)_{qr}$, 0_{qr} being the truncated simplex

$$t_q\, \alpha_{q+r+1} = t_r\, \alpha_{q+r+1}$$

(COXETER 1978, p. 131). The same symbol p_{qr} was later recognized as a natural abbreviation for a decorated Coxeter-Dynkin diagram. For instance, 5_{21} is the eight-dimensional honeycomb



(COXETER 1988, p. 34).

7. PROJECTIVE n-SPACE AND EUCLIDEAN $(9-n)$-SPACE

Gosset's polytopes had already been rediscovered in 1910 by ELTE (1912). SCHOUTE (1910) recognized one of them (Elte's V_{27}, my 2_{21}) as representing the cubic surface in such a way that the 27 vertices, 216 edges and 135 diagonals of the polytope correspond to the 27 lines, 216 skew pairs and 135 intersecting pairs on the surface. This remarkable correspondence was rigorously explained by TODD (1932). Figure 1 (drawn by Peter McMullen) shows the most symmetrical two-dimensional projection of the six-dimensional polytope. Its 27 vertices appear as 12 + 12 + 3; two concentric dodecagons with the remaining three vertices all projected into the centre (COXETER 1940, pp. 461-463; see also EDGE 1970, p. 757).

Since *skew* lines on the cubic surface F_2^3 correspond to *adjacent* vertices of 2_{21}, the lines on F_2^3 that are skew to one of them correspond to those vertices of 2_{21} which are adjacent to one vertex. In other words, the lines on F_2^4 correspond to the vertices of the vertex figure of 2_{21}, which is the *hemi-cube* $1_{21} = h\gamma_5$ (Elte's HM_5). Continuing, we see that the lines on F_2^n correspond to the vertices of $(5-n)_{21}$, for $n<7$.

In detail, 1_{21} has the 16 vertices

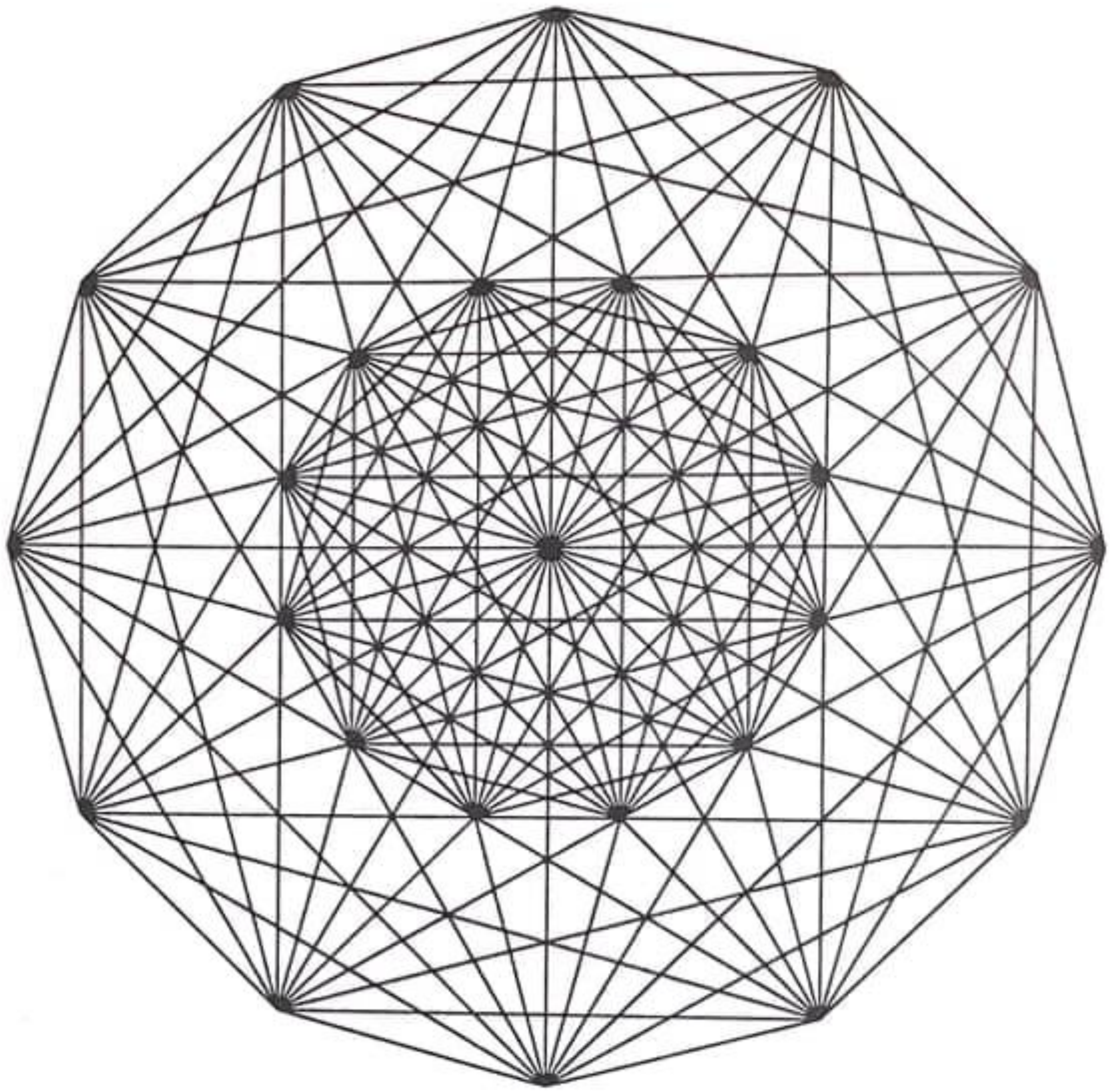


FIGURE 1. The six-dimensional polytope 2_{21}

$$(0, 0, 0, 0, 0)$$

$$(0, 1, 1, 1, 1)\quad (1, 0, 1, 1, 1)\ \ldots\ .(1, 1, 1, 1, 0),$$

$$(1, 1, 0, 0, 0)\quad (1, 0, 1, 0, 0)\ \ldots\quad (0, 0, 0, 1, 1)$$

(with an even number of ones), and these represent the $1 + 5 + 10$ lines

$$\begin{array}{cccc} & a_6 & & \\ b_1 & b_2 & \cdots & b_5 \\ c_{12} & c_{13} & \cdots & c_{45} \end{array}$$

on the quartic surface F_2^4. Similarly, the vertex figure of 1_{21} is the truncated 5-cell

$$0_{21} = \mathrm{t}_1\alpha_4$$

(Elte's $\mathrm{t}C_5$), whose 10 vertices

$$(1,1,0,0,0) \quad (1,0,1,0,0) \quad \cdots \quad (0,0,0,1,1),$$

the midpoints of the edges of the four-dimensional simplex

$$(2,0,0,0,0) \quad (0,2,0,0,0) \quad \cdots \quad (0,0,0,0,2),$$

represent the 10 lines

$$c_{12} \quad c_{13} \quad \cdots \quad c_{45}$$

on the quintic surface F_2^5. And the vertex figure of 0_{21} is the triangular prism

$$(-1)_{21} = \alpha_2 \times \alpha_1$$

(DU VAL 1933, p. 33), whose 6 vertices

$$\begin{matrix} (1,0,0,1,0) & (0,1,0,1,0) & (0,0,1,1,0) \\ (1,0,0,0,1) & (0,1,0,0,1) & (0,0,1,0,1) \end{matrix}$$

represent the lines

$$\begin{matrix} c_{14} & c_{24} & c_{34} \\ c_{15} & c_{25} & c_{35} \end{matrix}$$

on the sextic surface F_2^6. The vertex figure of $(-1)_{21}$ is the isosceles triangle

$$\begin{matrix} & (0,\ 1,\ 0,\ 1,\ 0) & \\ (1,\ 0,\ 0,\ 0,\ 1) & & (0,\ 0,\ 1,\ 0,\ 1) \end{matrix}$$

whose 3 vertices represent the lines

$$\begin{matrix} & c_{24} & \\ c_{15} & & c_{35} \end{matrix}$$

on F_2^7. Finally, at distance $\sqrt{2}$ from (1, 0, 0, 0, 1) we have only (0, 0, 1, 0, 1), and from (0, 1, 0, 1, 0) we have none, agreeing with del Pezzo's discovery of two distinct kinds of F_2^8.

Proceeding in the opposite direction, we would like to utilize the fact that 2_{21} is the vertex figure of the seven-dimensional polytope 3_{21} (COXETER 1988, p. 32) and hence to describe the 56 lines on some kind of two-dimensional 'del Pezzo surface' F_2^2. C.F. GEISER (1869, p. 129; DICKSON 1916, p. 351) observed that the enveloping cone from an arbitrary point P on the cubic surface intersects an arbitrary plane in a quartic curve whose 28 bitangents arise from the 27 lines, along with the tangent plane at P. DU VAL (1933, p. 57) interpreted this result as meaning that the cubic surface F_2^3 is projected from P onto a

'Geiser surface' F_2^2 which consists of a repeated plane, branching along the quartic curve whose 28 bitangents count as $\binom{8}{2}$ line pairs

$$c_{12}C_{12},\quad c_{13}C_{13},\ \cdots,\quad c_{78}C_{78}.$$

The line pair $c_{\mu\nu}C_{\mu\nu}$ $(\mu<\nu<7)$ arises from the line $c_{\mu\nu}$ on F_2^3, $c_{\nu 7}C_{\nu 7}$ from a_ν, $c_{\nu 8}C_{\nu 8}$ from b_ν, and $c_{78}C_{78}$ from the tangent plane at P. The two coincident lines $c_{\mu\nu}$ and $C_{\mu\nu}$ are regarded as intersecting each other twice: at the two points of contact of this bitangent with the quartic curve. Pairs that intersect just once are $c_{12}C_{13}, c_{12}c_{34}, C_{12}C_{34}$, etc., while skew pairs are $c_{12}c_{13}, C_{12}C_{13}, c_{12}C_{34}$, etc.. In particular, the lines skew to C_{78} are $c_{\mu\nu}$ $(\mu<\nu<7)$, $C_{\nu 7}$, $C_{\nu 8}$; these are projections of the lines $c_{\mu\nu}$, a_ν, b_ν on F_2^3.

We see from the coordinates (Coxeter 1988, pp. 8, 25, 28) that, in terms of the edge-length as unit, the polytope p_{21} for $p \leqslant 3$ has diagonals of length $\sqrt{2}$, while 3_{21} (Coxeter 1928) has also *diameters* of length $\sqrt{3}$. Thus, in every case with $2 \leqslant n \leqslant 6$ and $m=1$ or 2 (or 3 if $n=2$);

Two vertices of the polytope $(5-n)_{21}$ at distance $\sqrt{m}$ represent two lines on the surface F_2^n having $m-1$ points of intersection.

In particular, as we have seen, any two of the 27 vertices of 2_{21} belong either to an edge $(m=1)$ or to a diagonal $(m=2)$, and any two of the 27 lines on F_2^3 are either skew or intersecting.

Taking successive vertex figures, we see that, since $E_6=[3^{2,2,1}]$ is both the symmetry group of 2_{21} and the automorphism group of the lines on F_2^3,

$[3^{5-n,2,1}]$ *is the symmetry group of* $(5-n)_{21}$

and *the automorphism group of the lines on* F_2^n.

In detail (Coxeter 1988, p. 17), the groups

$$E_7,\quad E_6,\quad E_5=D_5,\quad E_4=A_4,\quad E_3=A_2\times A_1$$

(mentioned at the end of §4) are the symmetry groups of the polytopes

$$3_{21},\quad 2_{21},\quad 1_{21}=\mathrm{h}\gamma_5,\quad 0_{21}=\mathrm{t}_1\alpha_4,\quad (-1)_{21}=\alpha_2\times\alpha_1$$

and the automorphism groups of the lines on the del Pezzo surfaces

$$F_2^2,\quad F_2^3,\quad F_2^4,\quad F_2^5,\quad F_2^6.$$

8. Diagrams containing circuits

The complete list of irreducible reflection groups, finite and Euclidean (Coxeter 1934, p. 619), has been quoted so often that there is no need to repeat it here. (Some confusion is caused by many authors' unfortunate habit of calling the *Euclidean* groups 'affine'.) One notices that the Coxeter-Dynkin diagram is a *tree* in every case except

$$[3^{[n]}]=\tilde{A}_{n-1},$$

where it is an n-gon (COXETER 1988, p. 5). In this case, if we work in the hyperplane $\Sigma u_\nu = 0$ of Cartesian n-space, the n generators R_ν are reflections in the hyperplanes

$$u_1 - u_2 = 0,\ u_2 - u_3 = 0, \cdots, u_{n-1} - u_n = 0,\ u_n - u_1 + 1 = 0:$$

each $R_\nu (\nu < n)$ transposes the coordinates u_ν and $u_{\nu+1}$, while R_n is the transformation

$$u'_1 = u_n + 1,\ u'_2 = u_2, \cdots,\ u'_{n-1} = u_{n-1},\ u'_n = u_1 - 1.$$

This infinite reflection group has an infinite normal subgroup generated by translations such as T^p, where

$$T = R_1 R_2 \cdots R_n R_{n-1} R_{n-2} \cdots R_2; \tag{8.1}$$

that is, T is the translation

$$u'_1 = u_1 + 1,\ u'_2 = u_2 - 1,\ u'_3 = u_3, \cdots, u'_n = u_n.$$

To symbolize its quotient group $G(p,p,n)$ of order $p^{n-1}n!$ (SHEPHARD 1953, p. 379; SHEPHARD and TODD 1954, p. 277; COXETER 1957, pp. 244, 248, 251), we decorate the Coxeter-Dynkin diagram by inserting the number p inside the n-gon. Just as in the case of $\tilde{A}_{n-1}$ itself, any two of the n generating involutions $R_1, \cdots, R_n$ are braided if their subscripts are consecutive *in cyclic order* and are commutative otherwise, and we have also the extra relation

$$T^p = 1. \tag{8.2}$$

Since T^p is the translation

$$u'_1 = u_1 + p,\ u'_2 = u_2 - p,\ u'_\nu = u_n\ (\nu > 2),$$

we may now regard the coordinates as residues modulo p. Thus when $p = 2$, the group is D_n, permuting the 2^{n-1} vertices of the hemi-cube $h\gamma_n$ (COXETER 1988, p.5), namely the points

$$(\pm 1, \pm 1, \cdots, \pm 1)$$

with an even number of minus signs. We may equally well take the coordinates to be homogeneous and replace R_n by

$$u'_1 = -u_n,\ u'_\nu = u_\nu\ (1 < \nu < n),\ u'_n = -u_1.$$

In other words, the Y-shaped diagram for D_n can be replaced by an n-gon with the number 2 written inside.

When $n = 3$, so that $T = R_1 R_2 R_3 R_2$, the diagram

for the octahedral group $D_3 = A_3$, which is isomorphic to the symmetric group of degree 4, means that the generators R_1, R_2, R_3 satisfy the presentation

$$R_r^2 = 1,\ \ R_1R_2R_1 = R_2R_1R_2,\ \ R_1R_3 = R_3R_1,\ \ R_2R_3R_2 = R_3R_2R_3.$$

Replacing R_3 by its conjugate $\overline{R}_3 = R_2R_3R_2$, we have

$$\overline{R}_3R_2\overline{R}_3 = R_2R_3R_2R_3R_2 = R_2\overline{R}_3R_2,$$

$$\overline{R}_3R_1\overline{R}_3 = R_3R_2R_1R_2R_3 = R_1R_3R_2R_3R_1 = R_1\overline{R}_3R_1$$

and the period of $R_1R_2\overline{R}_3R_2 = R_1R_3$ is 2. Hence

In terms of fundamental regions (Coxeter 1990, pp. 16, 20), the spherical triangle (3 3 2) is combined with one of its neighbours to form the larger triangle $(33\frac{3}{2})$, as in Figure 2 (Davis, Grünbaum and Sherk 1981, p. 163, where B should be D).

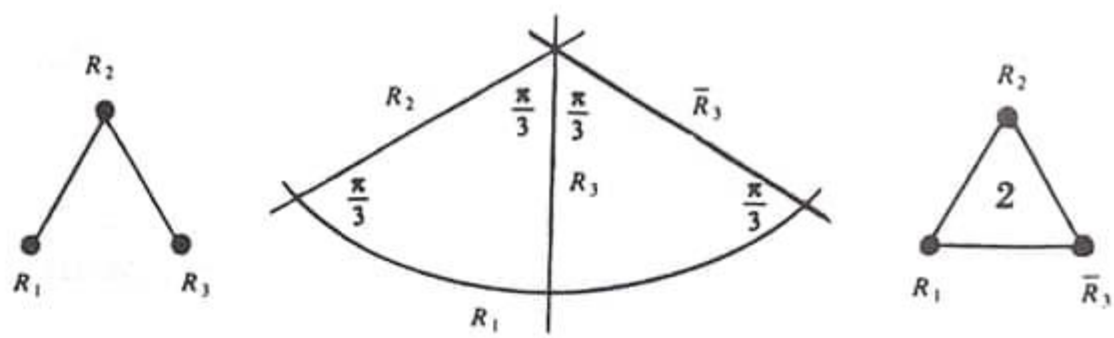
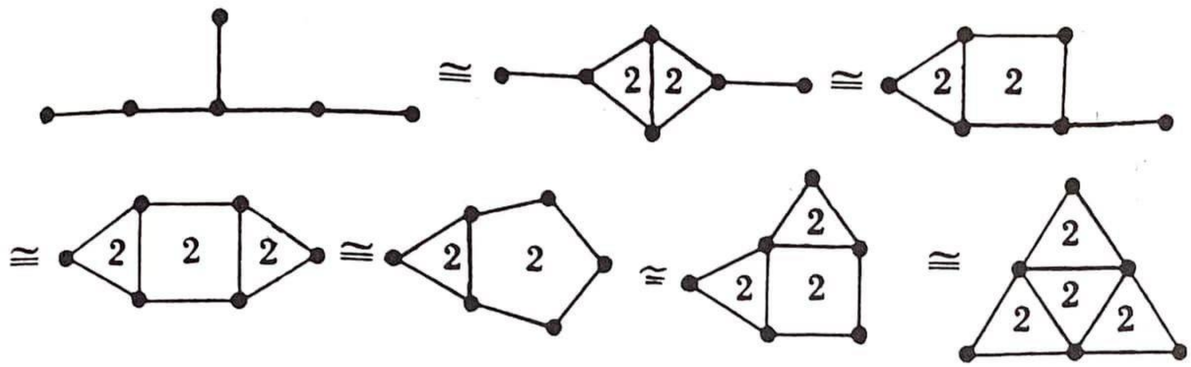


Figure 2. The triangle $(3\,3\frac{3}{2})$ dissected in two two triangles (3 3 2)

9. Exotic presentations

In the same spirit, whenever two generating involutions A and B are braided, we can replace one of them by $ABA\,(=BAB)$ so as to obtain a possibly different presentation for the same group. In particular, we can add a tail (or two or three tails) of any length, to a triangle marked 2, and the group will still be symmetric (Coxeter 1957, p. 250). Similarly, D_n can be presented as an n-gon marked 2 or, if $n > 4$, as an m-gon marked 2 with $m < n$ and one tail of length $n - m$. In particular, D_4 can be presented as a square (marked 2), or by a pair of triangles, both marked 2, with a common side.

For E_6 we obtain, in succession, the presentations



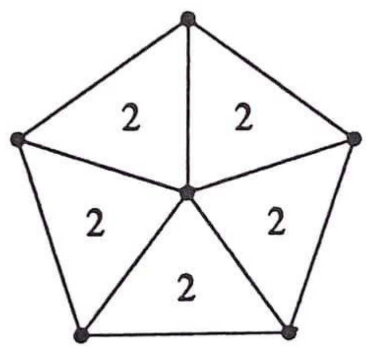
In the last case, the generating reflections may be taken to be

N_{156}

N_{12} N_{13}

N_{246} N_{23} N_{345}

A still more symmetrical presentation for E_6, suggested by TSARANOV (1989), is



with generators

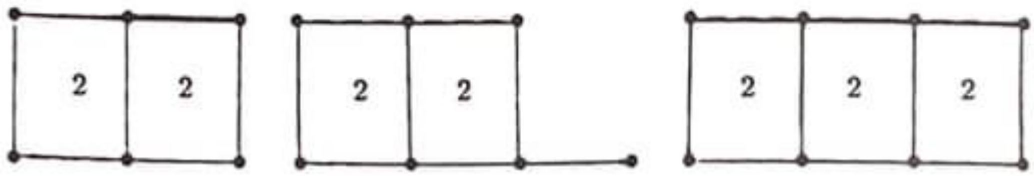
N_{126}

N_{236} N_{156}

N

N_{346} N_{456}

For E_6, E_7, E_8, Peter McMullen found many other exotic presentations, including



which were almost anticipated by CARTER (1972, pp. 10, 11; see also SCHELLEKENS and WARNER (1988).

Finally, extended diagrams for hyperbolic reflection groups gave been used by DU VAL (1933, p. 73), VINBERG (1971, p. 1084), CONWAY and SLOANE (1988, pp. 529, 570) and many others. It was ARNOLD (1974, p. 20) who remarked that 'the classification of more complex singularities provides new wonderful coincidences, where Lobachevsky triangles and automorphic functions take part.'

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