Orthogonal Trees

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1. Introduction

Any *tree*, with n edges and $n+1$ vertices, can be realized in Euclidean n-space so that its edges, of any chosen lengths, are *mutually perpendicular* (Coxeter 1989, p.60). The convex hull of such an *orthogonal tree* is an *orthogonal simplex* whose dihedral angles include $\binom{n}{2}$ right angles. More precisely, each vertex of the tree, being also a vertex of the simplex, represents (as in a Coxeter-Dynkin diagram) the opposite facet of the simplex. The two ends of an edge of the tree represent two facets forming an acute dihedral angle; each of the remaining $\binom{n}{2}$ pairs of facets are orthogonal. This happens because, for any two non-adjacent vertices of the tree, the minimal subgraph joining them determines a simplex (the *orthoscheme* of Coxeter 1973, p.137) whose first and last facets are orthogonal. Since the remaining edges of the tree are orthogonal to the subspace spanned by the orthoscheme, these "first and last facets" are sections of orthogonal facets of the whole n-simplex.

2. Edges and Altitudes

Let $P_0, P_1, \ldots, P_n$ be the vertices (in any order) of an orthogonal tree; let

$$\ell_{\mu\nu} = P_\mu P_\nu$$

(for various values of $\mu \neq \nu$) be the lengths of its n mutually orthogonal edges; and let $a_0, a_1, \ldots, a_n$ be the *altitudes* of the simplex, so that a_ν is the distance from P_ν to the opposite facet. (It may happen that $\ell_{\mu\nu} = a_\mu$ or a_ν.) Finally, let $\alpha_{\mu\nu}$ be the dihedral angle opposite to the edge $P_\mu P_\nu$ (which belongs to the tree), that is, the acute angle between the facets opposite to P_μ and P_ν. Since these are *adjacent* vertices of the tree, there are n such angles. Since the edges of the tree are mutually orthogonal, the remaining $\binom{n}{2}$ dihedral angles of the n-simplex are right angles.

We proceed to establish the surprisingly simple formula

$$\cos \alpha_{\mu\nu} = a_\mu a_\nu / \ell_{\mu\nu}^2 \tag{2.1}$$

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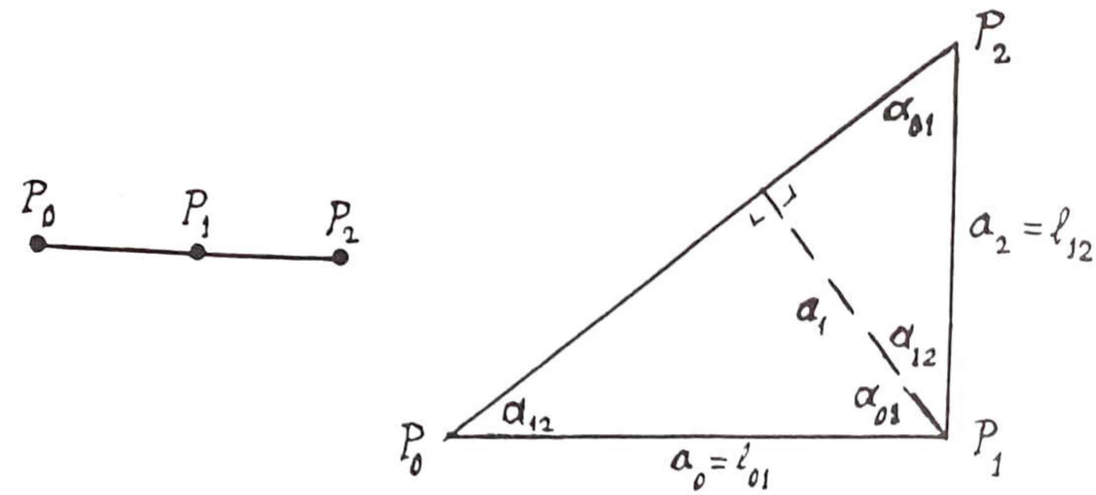


Figure 1: The right-angled triangle

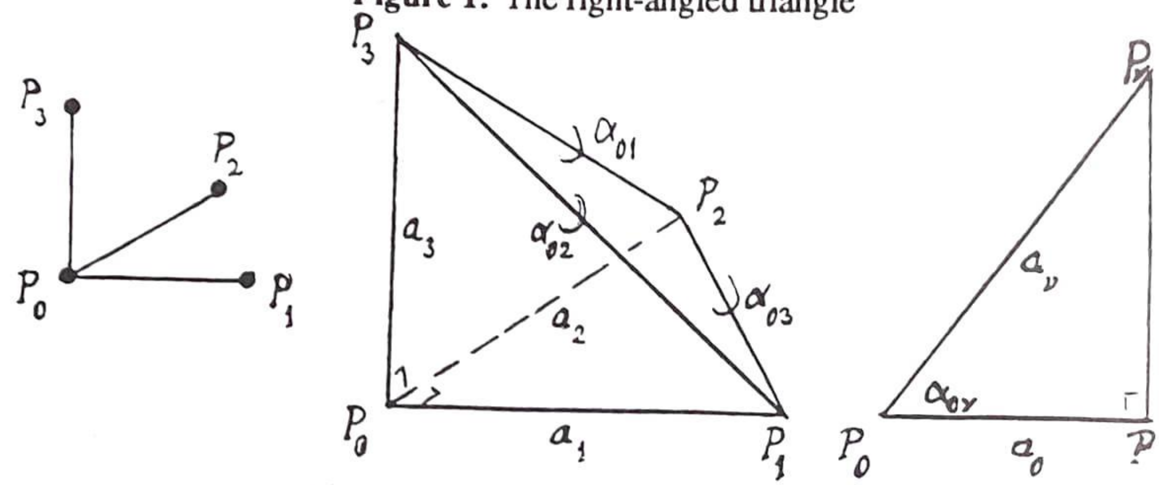


Figure 2: The trirectangular tetrahedron

If the vertex P_ν of the tree has valency 1, so that the altitude from P_ν coincides with the edge $P_\mu P_\nu$ and $\ell_{\mu\nu} = a_\nu$, the formula (2.1) reduces to

$$\cos \alpha_{\mu\nu} = a_\mu / a_\nu \tag{2.2}$$

and the section of the orthogonal n-simplex by the plane $a_\mu a_n u$ is an orthogonal 2-simplex, that is, a right-angled triangle (see Figure 1). Obviously

$$\cos \alpha_{01} = a_1/a_0, \quad \cos \alpha_{12} = a_1/a_2.$$

For instance, if the edges are $P_0 P_1, P_0 P_2, \dots, P_0 P_n$, like the axes for n-dimensional coordinates, we have

$$\ell_{0\nu} = a_\nu \qquad (\nu = 1, 2, \dots, n)$$

and the angle $\alpha_{0\nu}$ between the hyperplanes opposite to P_0 and P_ν is equal to the angle $P_\nu P_0 P$, where $P_0 P$ is the altitude a_0. (See Figure 2 for the case $n = 3$.)

Hence

$$\cos \alpha_{0\nu} = P_0 P / P_0 P_\nu = a_0 / a_\nu,$$

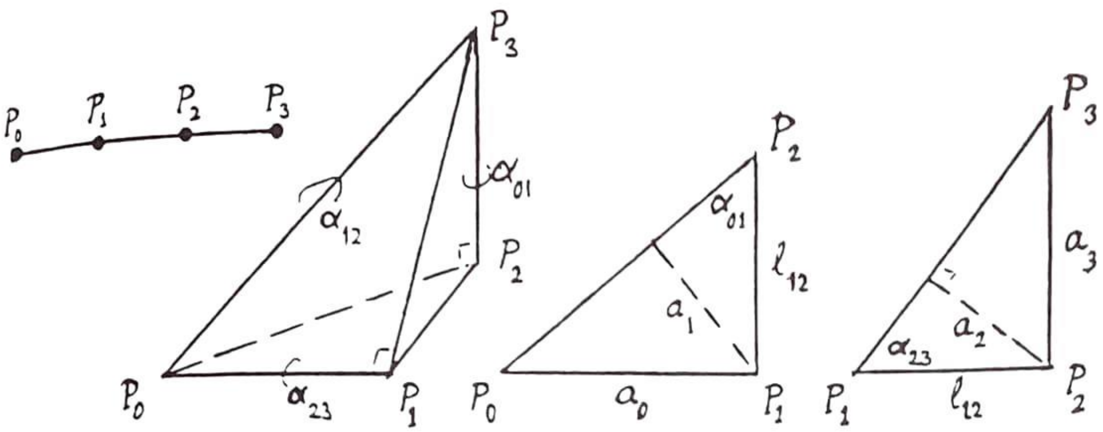


Figure 3: The quadrirectangular tetrahedron or 3-orthoscheme

in agreement with (2.2).

On the other hand, when the altitudes a_μ and a_ν are distinct from the edge $\ell_{\mu\nu}$, they span a 3-space, and the section of the orthogonal n-simplex by this 3-space is a 3-dimensional *orthoscheme* like $P_0P_1P_2P_3$ in Figure 3. The edges P_0P_1, P_1P_2, P_2P_3 are mutually orthogonal, and the dihedral angle α_{12} plays the role of $\alpha_{\mu\nu}$ in (2.1). It is known (Coxeter 1989, p.67(15)) that

$$\cos\alpha_{12} = \sin\alpha_{01}\sin\alpha_{23},$$

and we see from the triangles $P_0P_1P_2$ and $P_1P_2P_3$ that

$$\sin\alpha_{01} = a_1/\ell_{12} \quad \text{and} \quad \sin\alpha_{23} = a_2/\ell_{12}.$$

Hence

$$\cos\alpha_{12} = a_1a_2/\ell_{12}^2,$$

in agreement with (2.1), which has thus been proved.

Another interesting formula is suggested by a remark of Günter Pickert. Since the "weights" $c_\nu = 1/a_\nu$ satisfy the equations

$$c_\nu = \sum_{\mu\neq\nu} c_\mu \cos\alpha_{\mu\nu}$$

(Coxeter 1988, p.11), and $\cos\alpha_{\mu\nu} = a_\mu a_\nu \ell_{\mu\nu}^{-2}$, we have

(2.3) $$a_\nu^{-2} = \sum_\mu \ell_{\mu\nu}^{-2}$$

summed over all those edges $P_\nu P_\mu$ *of the tree* which emanate from the vertex P_ν.

Combining (2.3) with (2.1), we now have the means to express all the dihedral angles of the orthogonal simplex in terms of the edges of the orthogonal tree.

3. Vectors along the Alititudes

Pickert has observed also that (2.3) can be obtained directly, without using (2.1). Since the vertices of the tree are named in an arbitrary order, there will be no loss of generality if we take P_μ and P_ν to be P_1 and P_0, and take the edges emanating from P_0 to be

$$P_0 P_\mu \qquad (1 \le \mu \le m \le n)$$

so that the vertex P_0 of the tree has valency m. Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ denote the vectors along these edges while $\mathbf{e}_{m+1}, \ldots, \mathbf{e}_n$ are vectors along the remaining edges of the tree, directed away from P_0.

Any point X in the hyperplane $P_1 \ldots P_n$ (opposite to P_0) may be reached by a vector

$$(3.1) \qquad \overrightarrow{P_0 X} = \sum_1^n t_\nu \overrightarrow{P_0 P_\nu}, \qquad \sum_1^n t_\nu = 1.$$

For any $\nu > m$ there exists, in the tree, a path to P_ν from some P_μ $(\mu \le m)$ along certain edges lying in that hyperplane. Hence each $\overrightarrow{P_0 P_\nu}$ with $\nu > m$ can be expressed as $\mathbf{e}_\mu$ $(\mu \le m)$ plus a certain sum of $\mathbf{e}_\lambda$s with $\lambda > m$, and so

$$(3.2) \qquad \overrightarrow{P_0 X} = \sum_1^n t'_\nu \mathbf{e}_\nu, \quad \sum_1^m t'_\nu = 1,$$

each t'_ν being a certain sum of ts.

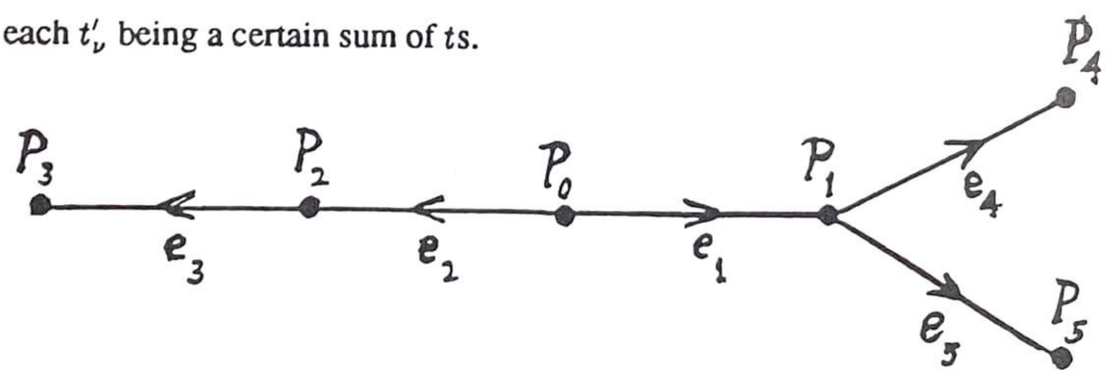


Figure 4: An example with $n = 5$ and $m = 2$.

For instance, in the case of the 5-dimensional tree indicated in Figure 4, we have

$$\begin{aligned} \overrightarrow{P_0 X} &= t_1 \overrightarrow{P_0 P_1} + t_2 \overrightarrow{P_0 P_2} + t_3 \overrightarrow{P_0 P_3} + t_4 \overrightarrow{P_0 P_4} + t_5 \overrightarrow{P_0 P_5} \\ &= t_1 \mathbf{e}_1 + t_2 \mathbf{e}_2 + t_3 (\mathbf{e}_2 + \mathbf{e}_3) + t_4 (\mathbf{e}_1 + \mathbf{e}_4) + t_5 (\mathbf{e}_1 + \mathbf{e}_5) \\ &= t'_1 \mathbf{e}_1 + t'_2 \mathbf{e}_2 + t'_3 \mathbf{e}_3 + t'_4 \mathbf{e}_4 + t'_5 \mathbf{e}_5 \end{aligned}$$

with $t'_1 = t_1 + t_4 + t_5$, $t'_2 = t_2 + t_3$ and $t'_\nu = t_\nu$ when $\nu > 2$, so that

$$t'_1 + t'_2 = t_1 + t_2 + t_3 + t_4 + t_5 = 1.$$

Although (3.2) holds for any point X in the hyperplane, something special happens when P_0X is the altitude a_0 (orthogonal to the hyperplane) so that, for all $\nu > m$, the vector $\mathbf{a}_0$ along this altitude satisfies

$$\mathbf{a}_0 \cdot \mathbf{e}_\nu = 0 \qquad (\nu > m.)$$

In this case (3.2) yields $t'_\nu = 0$ for all $\nu > m$, and

(3.3)
$$\mathbf{a}_0 = \sum_1^m t'_\mu \mathbf{e}_\mu, \quad \sum_1^m t'_\mu = 1.$$

If $m = 1$, we have simply $\mathbf{a}_0 = \mathbf{e}_1$; otherwise $\mathbf{a}_0$ may be described as the perpendicular from P_0 to the $(m-1)$-space $P_1 \ldots P_m$.

For any $\lambda < m$, the vector $\mathbf{e}_\lambda - \mathbf{e}_m$ lies in this $(m-1)$-space and

$$\mathbf{a}_0 \cdot (\mathbf{e}_\lambda - \mathbf{e}_m) = 0, \quad \mathbf{a}_0 \cdot \mathbf{e}_\lambda = \mathbf{a}_0 \cdot \mathbf{e}_m$$

(the same for all λ). Since $\mathbf{e}_\lambda^2 = \ell_{0\lambda}^2$, (3.3) yields

$$t'_\lambda \ell_{0\lambda}^2 = t'_m \ell_{0m}^2 = k,$$

say. So $t'_\mu = k\ell_{0\mu}^{-2}$. But

$$1 = \sum_1^m t'_\mu = k \sum_1^m \ell_{0\mu}^{-2}.$$

Therefore

$$k^{-1} = \sum_1^m \ell_{0\mu}^{-2}, \quad \mathbf{a}_0 = \sum_1^m t'_\mu \mathbf{e}_\mu = k \sum_1^m \ell_{0\mu}^{-2} \mathbf{e}_\mu,$$

$$a_0^2 = \mathbf{a}_0^2 = k^2 \left(\sum_1^m \ell_{0\mu}^{-2} \mathbf{e}_\mu \right)^2 = k^2 \sum_1^m \ell_{0\mu}^{-4} \ell_{0\mu}^2 = k^2 k^{-1} = k$$

and

(3.4)
$$a_0^{-2} = \sum_1^m \ell_{0\mu}^{-2}.$$

Thus (2.3) has been proved a new way.

Since $a_0^2 = k$, the unit vector along the altitude a_0 from P_0 is

$$a_0^{-1}\mathbf{a}_0 = a_0 \sum_1^m \ell_{0\mu}^{-2}\mathbf{e}_\mu .$$

Similarly, the unit vector along the altitude a_1 (of the same simplex $P_0P_1 \ldots P_n$) from P_1 is

$$a_1^{-1}\mathbf{a}_1 = a_1(\ell_{10}^{-2}\mathbf{e}'_0 + \sum_2^{m'} \ell_{1\mu}^{-2}\mathbf{e}'_\mu),$$

where $\mathbf{e}'_1, \mathbf{e}'_2, \ldots$ are the m' edges of the tree emanating from P_1. Since $\ell_{10} = \ell_{01}$ and

$$\mathbf{e}'_0 = -\mathbf{e}_1$$

and all the other $\mathbf{e}'_\mu$ are orthogonal to all the other $\mathbf{e}_\mu$, we have

$$\cos(\pi - \alpha_{01}) = a_0^{-1}\mathbf{a}_0 \cdot a_1^{-1}\mathbf{a}_1 = -a_0a_1\ell_{01}^{-4}\ell_{01}^2 = -a_0a_1\ell_{01}^{-2}.$$

Thus

(3.5) $$\cos\alpha_{01} = a_0a_1\ell_{01}^{-2}$$

and (2.1) has been proved a new way.

4. The Edge Lengths of Some Important Simplexes

It is known (Coxeter 1973, pp.190–194; Bourbaki 1968, p.199) that each of the irreducible infinite reflection groups in Euclidean n-space has, for its fundamental region, a *simplex*. It happens, in the cases

$$\bar{B}_n, \quad \bar{C}_n, \quad \bar{D}_n, \quad \bar{E}_6, \quad \bar{E}_7, \quad \bar{E}_8, \quad \bar{F}_4, \quad \bar{G}_2,$$

(i.e., in every case except $\bar{A}_n$), that this is an *orthogonal* simplex. (In Coxeter 1973, p.194, these eight cases are denoted by

$$S_{n+1}, \quad R_{n+1}, \quad Q_{n+1}, \quad T_7, \quad T_8, \quad T_9, \quad U_5, \quad V_3.)$$

According to the "crystallographic restriction" (see, e.g., Coxeter 1973, p.63), the dihedral angles of such a fundamental region can only be of the form π/q, where $q = 2, 3, 4$, or 6. These angles are known in each case, as also are the altitudes a_ν, because they are inversely proportional to the "weights" z^ν or c_ν (Coxeter 1973, p.183) which are directly proportional to the contents of the simplex's facets. To compute the edge lengths in terms of the *longest* edge of the tree as a unit, we write $1/c_\nu$ for a_ν in (2.1) so as to obtain

$$\ell_{\mu\nu} = (c_\mu c_\nu \cos \alpha_{\mu\nu})^{-\frac{1}{2}}. \tag{4.1}$$

The results are exhibited in the accompanying table.

In these trees, each edge indicates a dihedral angle $\alpha_{\mu\nu} = \pi/3$ of the orthogonal simplex, except when the edges is double ($\alpha_{\mu\nu} = \pi/4$) or triple ($\alpha_{\mu\nu} = \pi/6$). In the cases $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, all the edges are single, so $\cos \alpha_{\mu\nu} = \frac{1}{2}$ and (4.1) reduces to

$$\ell_{\mu\nu} = (\frac{1}{2} c_\mu c_\nu)^{-\frac{1}{2}}.$$

This simple formula was discovered by Edward Pervin, whose letter of 1990 to the author inspired the present investigation.

Of course, (4.1) refers only to the n *mutually orthogonal* edges; the rest are easily obtained (by Pythagoras) as square roots of sums of squares. For instance, the longest edge of the simplex for $\tilde{E}_8$ is the square root of

$$1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{12} + \frac{1}{4} = 2.$$

This ($\sqrt{2}$) is also the longest edge of a 7-orthoscheme which is a facet of the 8-simplex. We must be careful not to confuse this Euclidean 7-orthoscheme with the spherical 7-orthoscheme

which is the fundamental region for the subgroup A_8 of $\tilde{E}_8$. Actually, the Euclidean 7-orthoscheme, with its vertices numbered from 0 to 7, has

$$c_5 = 3\sqrt{3} \qquad \text{(instead of 6)}$$

and

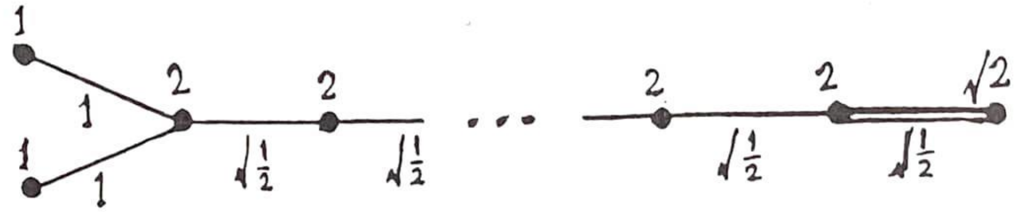
$$\alpha_{45} = \alpha_{56} = \arccos \sqrt{\frac{1}{3}} \qquad \text{(instead of } \pi/3.)$$

5. The Isohedral Simplex for $\tilde{A}_n$

For the sake of completeness, let us compute the edge lengths of the fundamental region for $\tilde{A}_n$, although this exceptional simplex has no set of n mutually orthogonal edges. According to Conway and Sloane (1988, p.460), the vertex V_ν of this simplex $V_0 V_1 \ldots V_n$ has ν coordinates $n+1-\nu$ followed by $n+1-\nu$ coordinates ν. Therefore

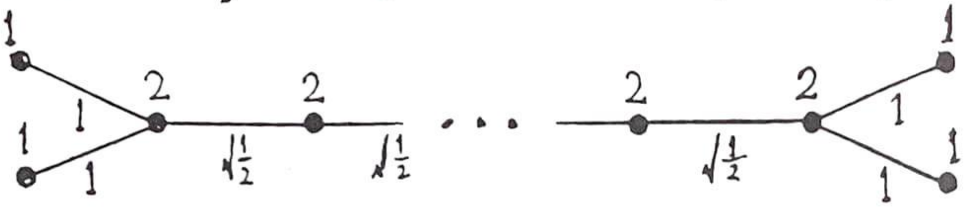
$$\begin{aligned} V_0 V_\nu^2 &= V_1 V_{\nu+1}^2 = \ldots \\ &= \nu(n+1-\nu)^2 + (n+1-\nu)\nu^2 \\ &= (n+1)\nu(n+1-\nu), \end{aligned}$$

$\tilde{B}_n$

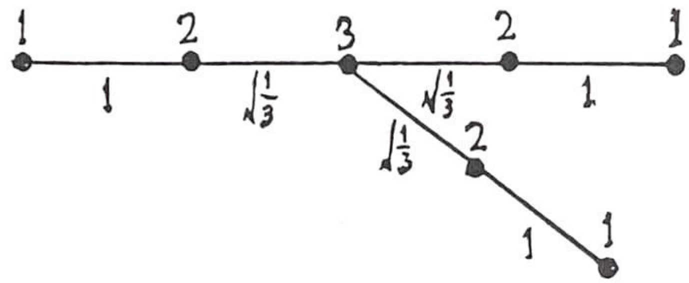


$\tilde{C}_n$

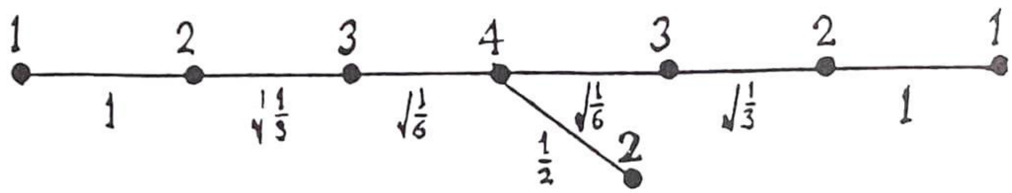
$\tilde{D}_n$



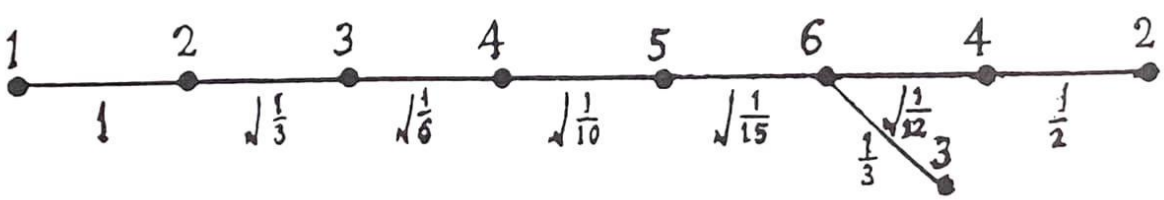
$\tilde{E}_6$



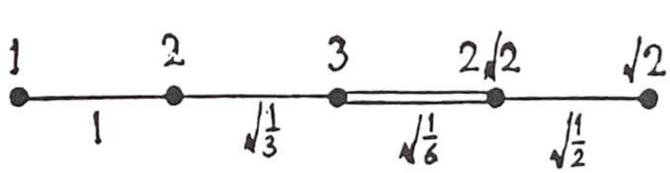
$\tilde{E}_7$



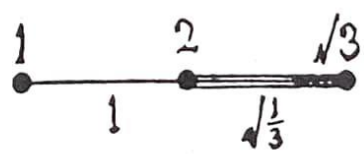
$\tilde{E}_8$



$\tilde{F}_4$



$\tilde{G}_2$



whence, after a change of scale,

$$V_0 V_\nu = \sqrt{\nu(n+1-\nu)}\,. \tag{5.1}$$

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