Regular and semi-regular polytopes. I.

By

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Contents.

Preface.

A "preliminary note" on this subject has appeared under the title *Wythoff's construction for uniform polytopes* [1]. I have divided the present work into three parts (of which the first appears now), the emphasis being on space of three dimensions, four dimensions, and six to eight dimensions, respectively. (Nothing special happens in five dimensions!) The treatment is elementary and *ab initio*; this inevitably involves giving an improved version of some earlier work. § 1. 8 is entirely new.

§ 1. 1.

The idea of a uniform polyhedron.

A study of the regular and Archimedean solids makes it seem desirable to invent a name for the class of polyhedra which have the following two properties:

(i) the faces are regular polygons,

isometries (ii) there is a group of ~~motions~~ (with or without reflections) which is transitive on the vertices.

[1] Proc. London Math. Soc. (2), 38 (1935), pp. 327—339.
= 'Twelve Geometric Essays,' Chapter 3.

Let us call these *uniform* polyhedra. In the present work we restrict consideration to polyhedra which are strictly convex. Kepler [2]) showed that the uniform polyhedra consist of:

(a) the five regular solids,

(b) the thirteen Archimedean solids (or semi-regular solids of the first kind),

(c) the prism, whose faces are two n-gons and n squares ($n = 3, 5, 6, \ldots$; we exclude $n = 4$ because the cube has already been counted).

(d) the antiprism, whose faces are two n-gons and $2\,n$ equilateral triangles ($n = 4, 5, 6, \ldots$; we exclude $n = 3$ because the octahedron has already been counted).

In all but two cases (the dodecahedron and the icosidodecahedron), the order of the group (ii) may be taken to be equal to the number of vertices. The edges of the polyhedron, suitably coloured, then constitute a "colour group" [3]). (See the Appendix.)

It is convenient to say that a polyhedron is *edge-reflexible* if all its edges are perpendicularly bisected by planes of symmetry. In such cases the group (ii) may be taken to be generated by reflections in these planes. For instance, the edges of the cube are bisected (in sets of four) by three planes, reflections in which generate a group of order 8. Again, the edges of the tetrahedron, and likewise of the octahedron, are bisected by six planes, reflections in which generate a group of order 24. (This time we are using a group whose order is greater than the number of vertices.)

It follows from the above definition that the vertices of any edge-reflexible polyhedron can be constructed as the set of transforms, under a group generated by reflections, of a single point. The actual cases may be very attractively exhibited by means of the polyhedral kaleidoscope invented by E. Hess [4]). This consists of a set of three plane mirrors, suitably inclined to one another. In a darkened room, the "single point" can be suggested by a candle flame; the vertices of the polyhedron are than seen as a multitude of such flames.

It happens that the only uniform polyhedra which are not edge-reflexible are the snub cube (Kepler's *cubus simus*), the snub dodecahedron, and the antiprisms. These can be treated somewhat similarly, using rotations instead of reflections. The two snub figures have no planes of symmetry at all, and consequently occur in *dextro* and *laevo* varieties.

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[2]) Opera omnia, **5** (Frankfurt, 1864), pp. 116—126. For a more rigorous treatment, see Catalan, Journal de l'École Polytech. **41** (1865), pp. 25—32.

[3]) Cayley, Proc. London Math. Soc. **9** (1878), pp. 126—133; Amer. Journal of Math. **11** (1889), pp. 139—157.

[4]) N. Jahrbuch für Mineralogie **1** (1889), pp. 54—65.

§ 1. 2.

Groups generated by reflections.

A single reflection generates a group of order two, denoted by [1]. Two reflections generate a group which is infinite unless the angle between their planes is commensurable with π. If the angle is $\frac{d\pi}{n}$ where d and n are coprime, the group is of order $2n$, denoted by $[n]$. In fact, the same group is generated by reflections in two planes making an angle $\frac{\pi}{n}$, and then the angular region between the planes serves as a fundamental region. The symbol $[\infty]$ naturally represents the infinite group which is generated by reflections in two parallel planes.

The group generated by reflections in any number of planes is equally well generated by reflections in these planes and all their transforms. If (as we shall always suppose) the group is discrete, the whole set of planes effects a partition of space into a finite or infinite number of congruent (or symmetric) regions, and the group is generated by reflections in the bounding planes of any one of the regions. Let these bounding planes be denoted by $p_1, p_2, \ldots$ (By an argument due to Pólya [5]) there cannot be more than six of them, in ordinary space.) Let $\frac{\pi}{n_{\mu\nu}}$ denote the (internal) angle between p_μ and p_ν. Then $n_{\mu\nu}$ is an integer, since otherwise the region would be subdivided by transforms of these planes. The case when p_μ and p_ν are parallel may be included by allowing $n_{\mu\nu}$ to be infinite.

Let R_μ denote the reflection in p_μ. Clearly

$$R_\mu^2 = 1, \quad (R_\mu R_\nu)^{n_{\mu\nu}} = 1. \tag{1.21}$$

By the following argument, due to Witt [6]), *the region enclosed by the planes* p_μ *is a fundamental region, and the above relations constitute an abstract definition*, the period of $R_\mu R_\nu$ being specified for every pair of the planes except such as are parallel. For instance, the dihedral group $[n]$ is defined by

$$R_1^2 = R_2^2 = (R_1 R_2)^n = 1,$$

while the analogous infinite group is the free product of two groups of order two:

$$R_1^2 = R_2^2 = 1.$$

[5]) Annals of Math. **35** (1934), p. 594.

[6]) Compare Burnside, Theory of Groups (second edition, Cambridge, 1911), p. 399 (§ 291); Cartan, Annali di Mat. (4) **4** (1927), pp. 215, 216.

Let W denote the region enclosed by the planes, so that WR_ν is the neighbouring region derived by reflecting in p_ν. A typical bounding plane of WR_ν is $p_\mu R_\nu$. The region derived from WR_ν by reflection in $p_\mu R_\nu$ is $WR_\mu R_\nu$, since this region and WR_ν, with their interface $p_\mu R_\nu$, are derived from WR_μ and W, with their interface p_μ, by applying the reflection R_ν. Similarly, the bounding plane $p_\lambda R_\mu R_\nu$ of this region leads to $WR_\lambda R_\mu R_\nu$. Thus the region WS, derived from W by applying any operation $S = R_\alpha R_\beta \ldots R_\mu R_\nu$, can be reached from W along a *path* which passes through the consecutively neighbouring regions

$$WR_\nu, WR_\mu R_\nu, \ldots, WR_\beta \cdots R_\mu R_\nu.$$

It remains to be proved that, if WS is W itself, then $S = 1$. In this case the path from W to WS is closed, and we consider what happens to the expression $R_\alpha R_\beta \ldots R_\mu R_\nu$ when this path is continuously shrunk to a point. If at any stage the path goes from one region into another and then immediately returns to the region it has just left, this detour may be eliminated by cancelling a repeated R_λ in the expression, in accordance with the relation $R_\lambda^2 = 1$. The only other kind of change that can occur during the shrinking process, is when the path momentarily crosses an *edge* (common to $2\,n_{\varkappa\lambda}$ regions). The expression will then be simplified by applying the relation $(R_\varkappa R_\lambda)^{n_{\varkappa\lambda}} = 1$.

The shrinkage of the path thus corresponds to an algebraic reduction of the expression $R_\alpha R_\beta \ldots R_\mu R_\nu$ by means of the above relations. Since the path can be shrunk right down to a point, the expression is equal to 1; and since this result is an algebraic consequence of (1. 21), those relations suffice for an abstract definition of the group.

An analogous argument proves the same result for any simply-connected space of l dimensions, the planes being replaced by hyperplanes, and the edges by $(l-2)$-spaces.

§ 1. 3.

The representation by graphs.

It is convenient to represent each generating reflection (or each mirror of the kaleidoscope) by a dot, and to draw a line (or "link") between two dots whenever the corresponding mirrors are not perpendicular, marking such a link with the number $n_{\mu\nu}$ to indicate the angle $\frac{\pi}{n_{\mu\nu}}$ $(n_{\mu\nu} \geqq 3)$. In this manner the group (1. 21) is represented by a *graph* of dots and (marked) links. Owing to its frequent occurrence, the mark 3 will usually be omitted (and left to be understood). Thus the group $[n]$ is denoted by

(1.31)

when $n = 1$,

when $n = 2$,

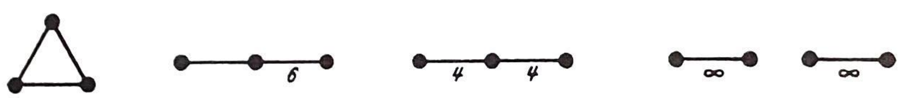
when $n = 3$,

when $n > 3$ (including $n = \infty$).

The disconnected symbol when $n = 2$ exhibits the fact that the group [2], generated by reflections in two perpendicular mirrors, is the direct product $[1] \times [1]$ (that is, the four-group).

When, as in these cases, all the mirrors are perpendicular to one plane, the group may be considered as operating in that plane, with one-dimensional mirrors. (The groups [1] and $[\infty]$ may even be regarded as operating in a line, with one or two *points* for mirrors). The remaining two-dimensional groups (which can be visualized by using three or four vertical mirrors) are infinite, and may be enumerated by considering, as possible fundamental regions, the various plane polygons whose angles are submultiples of π. There are four such groups with finite fundamental regions:

(1.32)



or, in a convenient alternative notation,

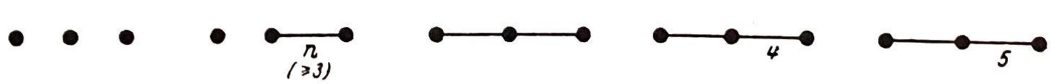
$$\Delta, \quad [3, 6], \quad [4, 4], \quad [\infty] \times [\infty].$$

(In the last case the fundamental region is a rectangle.) The group 

or $[1] \times [\infty]$ is not interesting, since it is infinite without having a finite fundamental region; in other words, it is an infinite group of a kind that has no finite analogue.

In all other cases with three mirrors, the planes are concurrent, and the group may be regarded as operating on a sphere centred at the point of concurrence; that is, we reflect in great circles of the sphere, and the fundamental region is a spherical triangle of angles $\frac{\pi}{l}, \frac{\pi}{m}, \frac{\pi}{n}$, where $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1$. We thus obtain the *extended polyhedral* groups:

(1.33)



or, in the other notation,

$$[1] \times [1] \times [1], \quad [1] \times [n], \quad [3, 3], \quad [3, 4], \quad [3, 5].$$

Of course $[m, n]$ is the same as $[n, m]$. Since the fundamental region is a spherical triangle of area $\frac{\pi}{2} + \frac{\pi}{m} + \frac{\pi}{n} - \pi$. the order is

$$\frac{4}{\frac{1}{m} + \frac{1}{n} - \frac{1}{2}}.$$

The network of such triangles covering the sphere is known as the *Dyck group-picture* [7]) for $[m, n]$.

We cannot use four or more concurrent mirrors, since the average angle of a spherical m-gon is greater than $\frac{m-2}{m}\pi$, whereas no angle of a fundamental region can be greater than $\frac{\pi}{2}$. But we may have four mirrors forming a wedge-shaped region, corresponding to the graph $\bullet\underset{n}{\text{——}}\bullet \quad \bullet\underset{\infty}{\text{——}}\bullet$. We may also take one (horizontal) mirror with three or four (vertical) mirrors all perpendicular to it; the graphs are then given by adding an isolated dot to each of (1. 32). These groups are, of course, infinite; but the really interesting infinite groups are those whose fundamental regions are finite polyhedra. Clearly, each dihedral angle of such a polyhedron must be a submultiple of π (namely $\frac{\pi}{2}$ or $\frac{\pi}{3}$ or $\frac{\pi}{4}$, etc.). The important thing to notice is that each angle is $\leqq \frac{\pi}{2}$.

Any polyhedron whose dihedral angles are all $\leqq \frac{\pi}{2}$ *is either a tetrahedron or a triangular prism or a rectangular parallelepiped.* To prove this, draw a sphere around any vertex of such a polyhedron, obtaining a spherical polygon whose angles are all $\leqq \frac{\pi}{2}$. We have seen that such a polygon can only be a triangle; hence the polyhedron has only trihedral vertices. Moreover, the sides of this triangle (regarded as lying on a sphere of unit radius) are face-angles of the polyhedron. But *if the angles of a spherical triangle are* $\leqq \frac{\pi}{2}$, *then also the sides are* $\leqq \frac{\pi}{2}$, since they are given by such formulae as

$$\cos c = \frac{\cos A \cos B + \cos C}{\sin A \sin B}.$$

Hence the face-angles are $\leqq \frac{\pi}{2}$, and the faces must be either triangles or rectangles. Now, if a polyhedron has only trihedral vertices, and its faces

[7]) W. Threlfall, Gruppenbilder. Abh. sächs. Akad. 41 Nr. 6 (1932), pp. 16, 26. For orthogonal projections of the actual cases, see Rouse Ball, Mathematical Recreations and Essays (eleventh edition, London 1939), p. 157. For stereographic projections, see Burnside, Theory of Groups (second edition, Cambridge 1911), pp. 405—407; Coxeter, Amer. Math. Monthly 45 (1938), pp. 522—525. When $\frac{1}{m} + \frac{1}{n} < \frac{1}{2}$, a group $[m, n]$ may be defined as operating in the hyperbolic plane, instead of on the sphere. The above formula for the order is no longer applicable; in fact it gives a negative value. The order is actually infinite, since the hyperbolic plane has an infinite area.

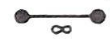
consist of F_3 triangles and F_4 rectangles ($F_3 \geqq 0$, $F_4 \geqq 0$), it must have $\frac{1}{2}(3F_3 + 4F_4)$ edges and $\frac{1}{3}(3F_3 + 4F_4)$ vertices, whence $\frac{1}{2}F_3$ and $\frac{1}{3}F_4$ are integers. By Euler's Theorem,

$$F_3 + F_4 - \tfrac{1}{6}(3F_3 + 4F_4) = 2,$$

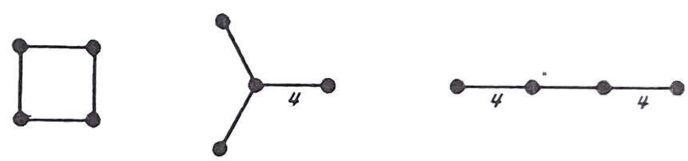
that is,

$$\tfrac{1}{2}F_3 + \tfrac{1}{3}F_4 = 2.$$

The solution $F_3 = 4$, $F_4 = 0$ gives a tetrahedron; $F_3 = 2$, $F_4 = 3$ gives a triangular prism; and $F_3 = 0$, $F_4 = 6$ gives a rectangular parallelepiped.

We obtain the groups with prismatic fundamental regions (including the rectangular parallelepiped) by adding the symbol  to each of (1. 32). On the other hand, if the fundamental region is a tetrahedron, each corner must form a trihedral angle of one of the types enumerated in (1. 33). In order to show that the only actual cases are

(1. 34)



or, in the other notation,

$$\square, \qquad \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}, \qquad [4, 3, 4],$$

$[3^{[4]}]$
$[4, 3^{1,1}]$

we may use Schläfli's formula

$$(1.35) \qquad \begin{vmatrix} 1 & -c_{12} & -c_{13} & -c_{14} \\ -c_{21} & 1 & -c_{23} & -c_{24} \\ -c_{31} & -c_{32} & 1 & -c_{34} \\ -c_{41} & -c_{42} & -c_{43} & 1 \end{vmatrix} = 0,$$

which relates the cosines ($c_{\mu\nu} = c_{\nu\mu}$) of the six dihedral angles of any tetrahedron.

Thorold Gosset has sent me the following elegant proof of this relation. Let A_1, A_2, A_3, A_4 denote the areas of the four faces (so that $c_{\mu\nu}$ is the cosine of the angle between the faces A_μ, A_ν). By projecting A_2, A_3, A_4 onto A_1, we obtain

$$A_1 = c_{12}A_2 + c_{13}A_3 + c_{14}A_4.$$

Thus

$$\begin{aligned} A_1 - c_{12}A_2 - c_{13}A_3 - c_{14}A_4 &= 0, \\ -c_{21}A_1 + A_2 - c_{23}A_3 - c_{24}A_4 &= 0, \\ -c_{31}A_1 - c_{32}A_2 + A_3 - c_{34}A_4 &= 0, \\ -c_{41}A_1 - c_{42}A_2 - c_{43}A_3 + A_4 &= 0. \end{aligned}$$

The desired result follows by eliminating the A_ν.

Precisely the same method of proof applies to the analogous formula for a simplex in Euclidean space of any number of dimensions. In two dimensions, as applied to a triangle ABC, it is equivalent to the formula

$$A + B + C = \pi.$$

§ 1. 4.

Abstract definitions for these groups and certain subgroups.

By (1. 21), the group $[m, n]$ or  has the abstract definition

(1. 41) $$R_1^2 = R_2^2 = R_3^2 = (R_1 R_2)^m = (R_1 R_3)^2 = (R_2 R_3)^n = 1.$$

Since each of these relations involves an even number of the R_ν, there is a subgroup of index two consisting of all products of even numbers of the R_ν. We denote this subgroup by $[m, n]'$; geometrically, it consists of all the *rotations* in $[m, n]$. It is generated by the two elements

$$S_1 = R_1 R_2, \quad S_2 = R_2 R_3,$$

which satisfy

(1. 411) $$S_1^m = S_2^n = (S_1 S_2)^2 = 1.$$

The notation $[m, n]'$ has been changed to $[m, n]^+$

These relations constitute a complete definition for $[m, n]'$, since (1. 41) can be derived from them by adjoining an involutory element R_2 which transforms S_1 and S_2 into their inverses, and then defining $R_1 = S_1 R_2$, $R_3 = R_2 S_3$. Similarly, the infinite group Δ or

(1. 42) $$R_1^2 = R_2^2 = R_3^2 = (R_1 R_2)^3 = (R_1 R_3)^3 = (R_2 R_3)^3 = 1$$

has the rotational subgroup Δ', defined by

(1. 421) $$S_1^3 = S_2^3 = (S_1 S_2)^3 = 1.$$

We observe that $[2, n]' \sim [n]$. This is the *dihedral* group; $[3, 3]'$, $[3, 4]'$, $[3, 5]'$ are the *tetrahedral*, *octahedral*, and *icosahedral* groups, respectively [8]).

When n is even, $[m, n]$ has another subgroup of index two, say $[m', n]$ or $[m^+, n]$, generated by S_1 and R_3. This is defined by

(1. 43) $$S_1^m = R_3^2 = (S_1^{-1} R_3 S_1 R_3)^{\frac{n}{2}} = 1;$$

for, $[m, n]$ can be derived from it by adjoining an involutory element R_1 (permutable with R_3) which transforms S_1 into its inverse, and then defining $R_2 = R_1 S_1$. We recognize $[3', 4]$ as the *pyritohedral* group; $[3', 6]$ and $[4', 4]$ are analogous infinite groups [9]).

[8]) W. Dyck, Math. Annalen **20** (1882), p. 34.

[9]) G. A. Miller, Amer. Journal of Math. **33** (1911), p. 368; Coxeter, Duke Math. Journal **2** (1936), pp. 66, 67.

When $m = 3$, the two elements $P = R_1 R_2 R_3$, $Q = R_3 R_2$ $(= S_2^{-1})$ generate $[3, n]$ itself in the form [10])

(1. 44) $$Q^n = (PQ)^2 = (P^2 Q^2)^3 = (P^3 Q^2)^2 = 1.$$

Since each of these relations involves P an even number of times, the implied period of P must be even. Let us denote it by h. We shall see (in § 1. 7) that $h = 4, 6, 10, \infty$ according as $n = 3, 4, 5, 6$. Since $P^2 = S_1 S_2^{-1} S_1^{-1} S_2$, $\frac{h}{2}$ is the period of the commutator of the generators of $[3, n]'$ [11]). When $n = 3$, (1.44) is equivalent to

$$P^4 = Q^3 = (PQ)^2 = 1,$$

which shows that

$$[3, 3] \sim [3, 4]'.$$

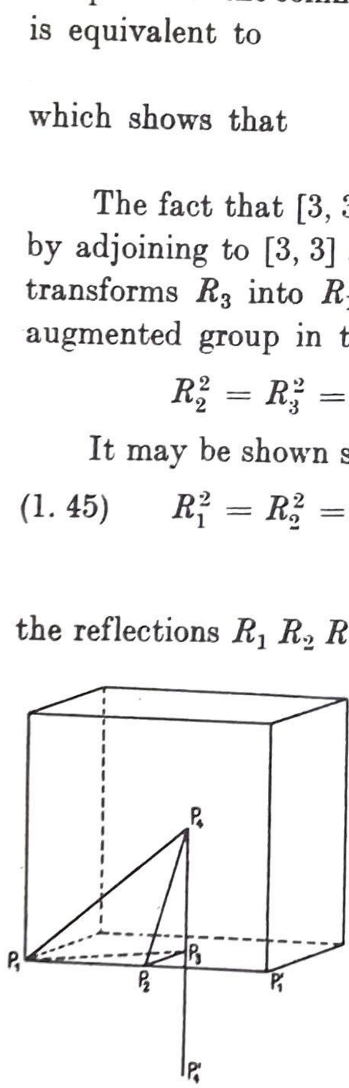
The fact that $[3, 3]$ is a subgroup of index two in $[3, 4]$ may also be seen by adjoining to $[3, 3]$ an involutory element R_4 (permutable with R_2) which transforms R_3 into R_1. On substituting $R_4 R_3 R_4$ for R_1, we obtain the augmented group in the form

$$R_2^2 = R_3^2 = R_4^2 = (R_2 R_3)^3 = (R_2 R_4)^2 = (R_3 R_4)^4 = 1.$$

It may be shown similarly that, in the infinite group $[4, 3, 4]$, defined by

(1. 45) $$R_1^2 = R_2^2 = R_3^2 = R_4^2 = (R_1 R_3)^2 = (R_1 R_4)^2 = (R_2 R_4)^2 = (R_1 R_2)^4 = (R_2 R_3)^3 = (R_3 R_4)^4 = 1,$$

the reflections $R_1 R_2 R_1$, R_2, R_3, R_4 generate a subgroup $\begin{bmatrix}3\\3\\4\end{bmatrix}$; in this, again,



the reflections $R_1 R_2 R_1$, R_2, R_3, $R_4 R_3 R_4$ generate a subgroup □. Thus the three groups (1. 34) are closely related.

Geometrically, $[4, 3, 4]$ is the complete symmetry-group of the ordinary cubic lattice; in fact, the fundamental region is a tetrahedron $P_1 P_2 P_3 P_4$ where P_1 is a vertex of one of the cubes, P_2 is the mid-point of an edge, P_3 is the centre of a square face, and P_4 is the centre of the whole cube. Let P_1' be another vertex of the cube, chosen so that P_2 is the mid-point of $P_1 P_1'$; and let P_4' be the centre of another cube, chosen so that P_3 is the midpoint of $P_4 P_4'$. Then $P_1 P_1' P_3 P_4$ is the fundamental region of the subgroup $\begin{bmatrix}3\\3\\4\end{bmatrix}$, and $P_1 P_1' P_4' P_4$ is that of the subgroup □.

10) Coxeter, Proc. Camb. Phil. Soc. **32** (1936), p. 195.

11) Coxeter, Trans. Amer. Math. Soc. **45** (1939), p. 88.

Each of these groups has its rotational subgroup of index two. Thus

$$R_1 R_2, \quad R_2 R_3, \quad R_3 R_4 \quad \text{generate}^{12)} \quad [4, 3, 4]',$$

$$(R_1 R_2)^2, \quad R_2 R_3, \quad R_3 R_4 \quad \text{generate} \quad \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}',$$

$$(R_1 R_2)^2, \quad R_2 R_3, \quad (R_3 R_4)^2 \quad \text{generate} \quad \square'.$$

The pyritohedral group may be said to have three infinite analogues, each a subgroup of index two in [4, 3, 4]:

$$R_1 R_2, \quad R_2 R_3, \quad R_4 \quad \text{generate} \quad [(4, 3)', 4],$$
$$R_1, \quad R_2 R_3, \quad R_4 \quad \text{generate} \quad [4, 3', 4],$$
$$R_1 R_4, \quad R_2, \quad R_3 \quad \text{generate} \quad [4', 3, 4'].$$ $= [(4,3,4,2^+)]$

In the last of these, the rotary reflections $R_2 R_1 R_4$, $R_1 R_4 R_3$ generate a subgroup which is denoted by[13] (4, 4 | 3, 3) because of its remarkably concise definition: $[(2^+, (4,3)^+, 4)]$

(1. 46) $$R^4 = S^4 = (RS)^3 = (R^{-1} S)^3 = 1.$$

Further groups, in which [4, 3, 4], [4, 3, 4]', [4, 3', 4], [4', 3, 4'], □, □' occur as subgroups of index two, can be derived by adjoining an involutory element T (geometrically, the rotation through π about the join of the mid-points of the edges $P_1 P_4$, $P_2 P_3$ of the tetrahedron $P_1 P_2 P_3 P_4$), which transforms R_1, R_2, R_3, R_4 into R_4, R_3, R_2, R_1. Thus

$$R_1, \quad T, \quad R_3 \quad \text{generate} \quad [[4, 3, 4]],$$
$$R_1 R_2, \quad T, \quad R_2 R_3 \quad \text{generate} \quad [[4, 3, 4]]',$$
$$R_1, \quad T, \quad R_2 R_3 \quad \text{generate} \quad [[4, 3', 4]],$$
$$R_1 R_4, \quad T, \quad R_2 \quad \text{generate} \quad [[4', 3, 4']] \quad \text{or}^{14)} \quad G^{4,6,6},$$ $[[(4,3,4,2^+)]]$
$$R_1 R_2 R_1, \quad T, \quad R_2 \quad \text{generate} \quad [\square],$$
$$(R_1 R_2)^2, \quad T, \quad R_2 R_3 \quad \text{generate} \quad [\square]'.$$

In [[4, 3, 4]], the rotary reflections $R_1 T$, $T R_3$ generate a subgroup (4, 6 | 2, 4), with a definition analogous to (1. 46)[15]. Similarly, the elements $R_1 R_2 R_1 T$, $T R_2$ of [□] generate (6, 6 | 2, 3). Finally, there are two interesting subgroups of index two in [[4', 3, 4']]:

$R_1 R_4$, $T R_2$ generate (2, 6, 6; 2), which is defined by

$$U^2 = V^6 = (UV)^6 = (UV^{-1} UV)^2 = 1;$$

and $R_1 R_4 T$, $T R_2$ generate the analogous group (2, 6, 4; 3).

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[12] See J. A. Todd, Proc. Camb. Phil. Soc. **27** (1931), p. 217, where these generators are called T_1, T_2, T_3.

[13] Coxeter, Trans. Amer. Math. Soc. **45** (1939), p. 81.

[14] Ibid., pp. 119, 125.

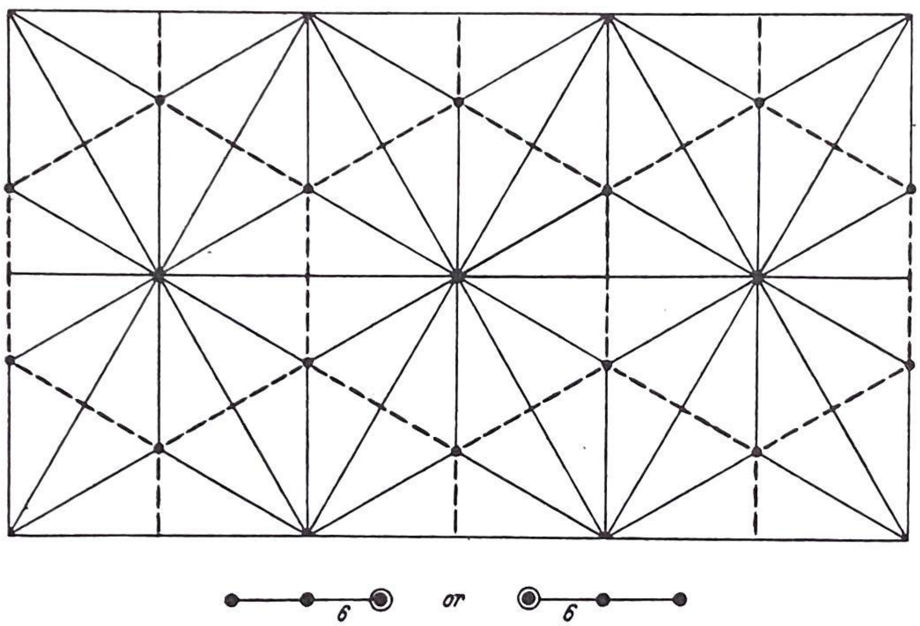
[15] Coxeter, Duke Math. Journal **2** (1936), p. 73.

We have now found generators for 20 of the 230 space-groups of crystallography. Continuing thus, we should be able to derive the rest of the 36 space-groups in the cubic system; but we have already gone far enough for our main purpose, which is to point the analogy to the finite groups in four dimensions.

§ 1. 5.

Wythoff's construction for uniform polyhedra.

Möbius [16]) drew several diagrams to show the set of all transforms of a point under an extended polyhedral group $[m, n]$. The points so obtained are the vertices of an edge-reflexible polyhedron, which is uniform if the initial point is suitably located. This construction was generalized in a very fruitful manner by W. A. Wythoff [17]); therefore I have named it after him.



The unextended group $[m, n]'$ leads similarly [18]) to a polyhedron which, though not in general edge-reflexible, is again uniform for a suitable position of the initial point. The infinite groups (1. 32) lead analogously to polygonal plane-fillings or *tessellations*, which may be regarded as infinite polyhedra [19]).

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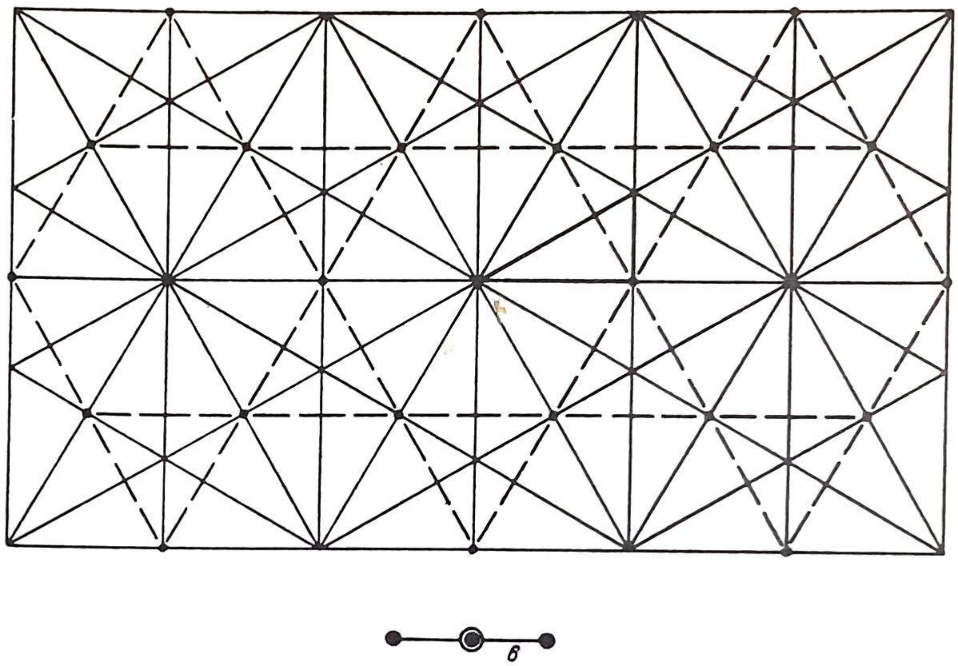
[16]) Gesammelte Werke **2** (1886), pp. 661, 677, 691.

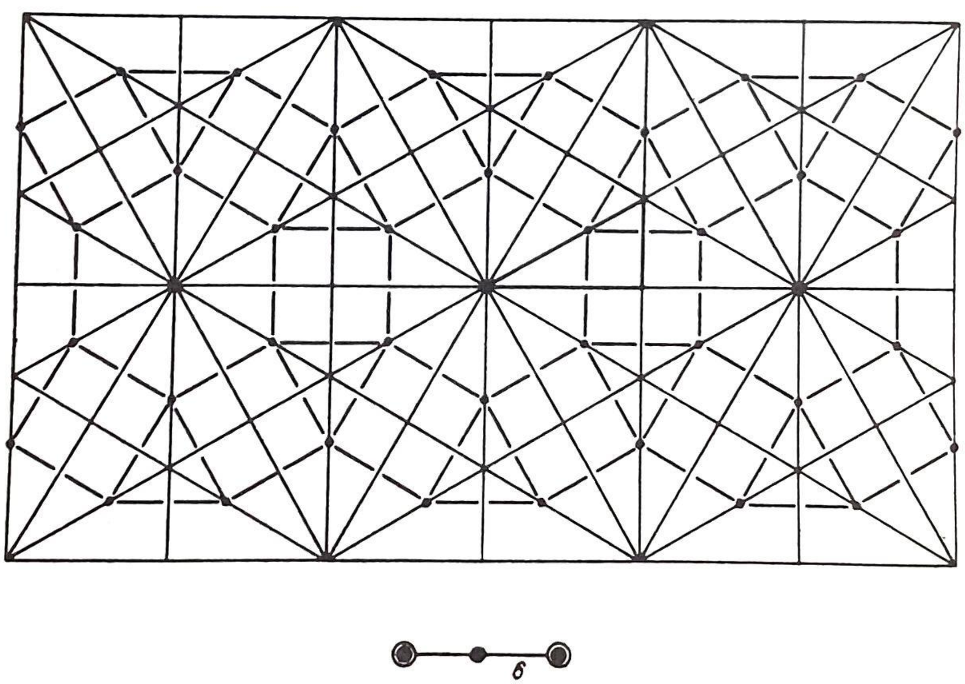
[17]) Proc. Royal Acad. of Sci., Amsterdam **20** (1918), pp. 966—970.

[18]) Möbius, loc. cit., pp. 656, 669, 688.

[19]) Kepler, Opera omnia, **5**, pp. 117—119 (Figs. D, E, F, L, N. P, S, V, Mm, Ii); Badoureau, Journal de l'École Polytech. **49** (1881), p. 93 (Figs. 61—66); Andreini, Mem. della Soc. ital. delle Sci. (3) **14** (1905), pp. 3—8 (Figs. 1—8, 9′, 10). Badoureau (through an error on p. 88) omitted Kepler's tessellation L or $3^4 . 6$ or $s\left\{\begin{matrix}3\\6\end{matrix}\right\}$, and was copied by several later authors.

The representation of the groups by graphs provides useful symbols for the polyhedra. The position of the initial point is indicated by drawing a ring around one or more of the dots in the graph. When the fundamental region is a (plane or spherical) triangle, the three dots originally symbolize its sides,



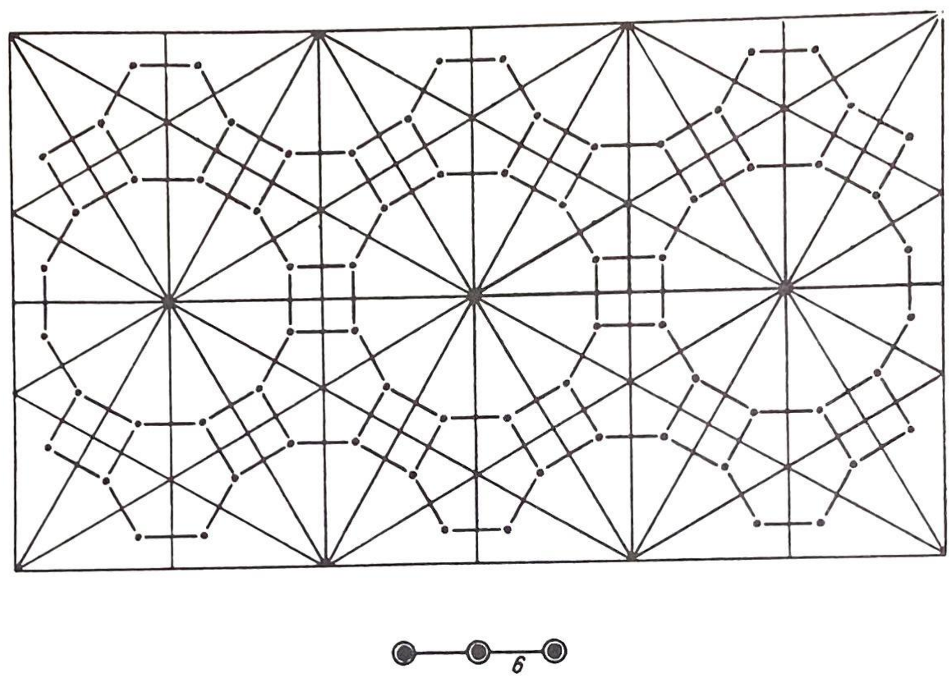


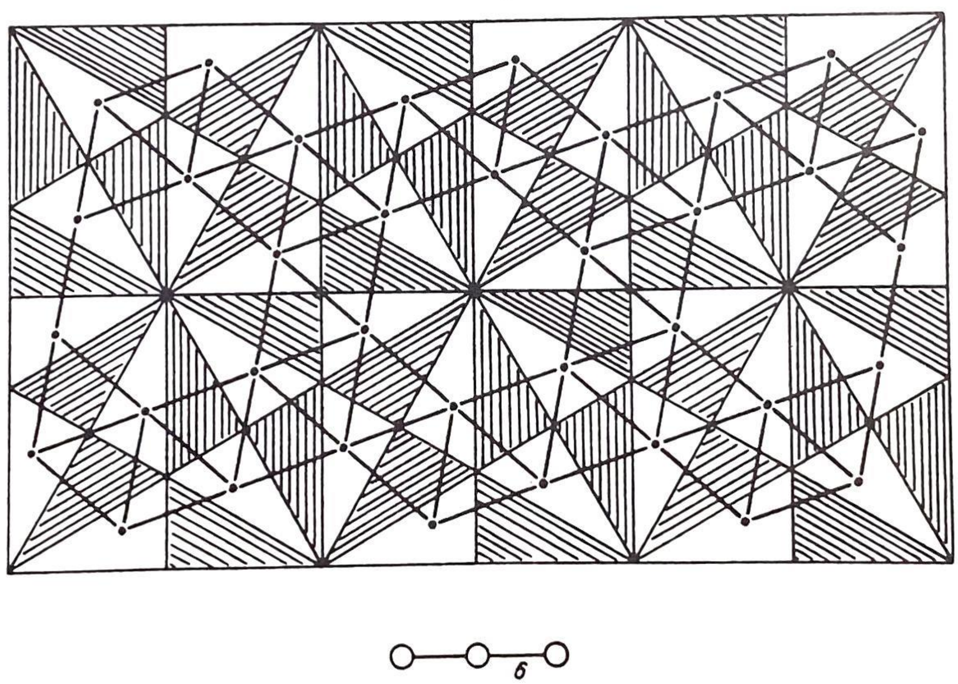
but can just as well be regarded as symbolizing the respectively opposite vertices.

In the following four illustrations, the fundamental region is a triangle of angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}$, and the derived (infinite) polyhedra are drawn in broken lines.

The polyhedron whose vertices are the transforms of a *vertex* of the triangle, is indicated by ringing the corresponding dot in the graph.

The polyhedron whose vertices are the transforms of a point lying on a *side* of the triangle, is indicated by ringing two dots. In order that the





edges of the polyhedron may be all equal, the chosen point must be equidistant from the two mirrors on which it does not lie; that is, its position, on the indicated side of the triangle, is determined as lying on the bisector of the opposite angle.

In both these cases, it is immaterial whether we use the group generated by reflections, or its rotational subgroup of index two.

The polyhedron whose vertices are the transforms, under the group generated by reflections, of a point lying *within* the triangle, is indicated by ringing all three dots. The chosen point now has to be equidistant from all three mirrors, i. e. at the in-centre of the triangle.

Finally, the polyhedron whose vertices are the transforms, under the rotational subgroup, of a point lying within the triangle, is indicated by removing the dots [20]) from this last symbol, leaving just the rings and the links. In this case the chosen point, say Q, is generally not at the in-centre, but is determined by the relations

$$\mathrm{AQ} \sin A = \mathrm{BQ} \sin B = \mathrm{CQ} \sin C$$

if the fundamental region ABC is a *plane* triangle, or

$$\sin \mathrm{AQ} \sin A = \sin \mathrm{BQ} \sin B = \sin \mathrm{CQ} \sin C$$

if it is a *spherical* triangle. We may recall, for the sake of comparison, that the corresponding relations for the in-centre, P, are

$$\mathrm{AP} \sin \frac{A}{2} = \mathrm{BP} \sin \frac{B}{2} = \mathrm{CP} \sin \frac{C}{2}$$

or

$$\sin \mathrm{AP} \sin \frac{A}{2} = \sin \mathrm{BP} \sin \frac{B}{2} = \sin \mathrm{CP} \sin \frac{C}{2}.$$

Thus Q cannot coincide with P unless

$$A = B = C \quad \left(= \frac{\pi}{3} \text{ or } \frac{\pi}{2}\right).$$

It is sometimes convenient to denote each uniform polyhedron by a symbol consisting of the numbers of sides of the various faces that surround a vertex, in their proper cyclic order. In this notation, the tessellations drawn above (in broken lines) are:

6. 6. 6, 3. 6. 3. 6, 3. 4. 6. 4, 4. 6. 12, 3. 3. 3. 3. 6.

The first two and the last of these are naturally abbreviated to

$$6^3, \quad (3.\,6)^2, \quad 3^4.\,6.$$

Similarly, the tessellations

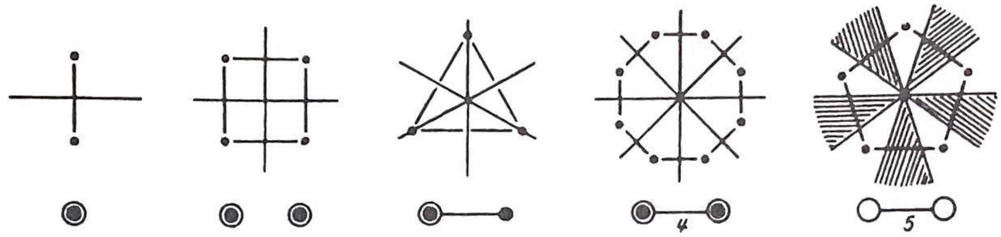
are

$$3^6, \quad (3.\,6)^2, \quad 6^3, \quad \text{and again} \quad 3^6.$$

20) This process can be thought of as "removing the reflections." This clever idea of using empty rings was proposed by Alicia Boole Stott.

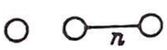
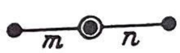
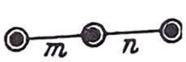
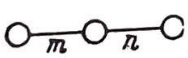
The analogous construction with only one or two mirrors (instead of three) leads to the symbols

for a line-segment, a square, an n-gon, a $2n$-gon, and again an n-gon ($n \geqq 3$).



The special virtue of this symbolism is that the symbol for a polyhedron contains the symbols for the various kinds of face.

The reader will have no difficulty in verifying that the groups $[m, n]$ and $[m, n]'$ lead to the following polyhedra:

	4^3	or	$\{4, 3\}$,	the cube;
	3^3	,,	$\{3, 3\}$,	the regular tetrahedron;
	$4^2 \cdot n$	,,	$\{\} \times \{n\}$	the n-gonal prism;
	$4^2 \cdot 2n$	,,	$\{\} \times \{2n\}$;	
	$3^3 \cdot n$	,,	$s\begin{Bmatrix} 2 \\ n \end{Bmatrix}$,	the n-gonal antiprism;
	m^n	,,	$\{m, n\}$,	the general regular polyhedron;
	$(m \cdot n)^2$	,,	$\begin{Bmatrix} m \\ n \end{Bmatrix}$	(described later);
	$(2m)^2 \cdot n$	,,	$t\{m, n\}$	the "truncated $\{m, n\}$";
	$m \cdot 4 \cdot n \cdot 4$	,,	$r\begin{Bmatrix} m \\ n \end{Bmatrix}$	the "rhombi-$\begin{Bmatrix} m \\ n \end{Bmatrix}$";
	$4 \cdot 2m \cdot 2n$	,,	$t\begin{Bmatrix} m \\ n \end{Bmatrix}$	the "truncated $\begin{Bmatrix} m \\ n \end{Bmatrix}$";
	$3^2 \cdot m \cdot 3 \cdot n$	,,	$s\begin{Bmatrix} m \\ n \end{Bmatrix}$.	

This list includes all the finite uniform polyhedra, with several repetitions, and all the uniform tessellations except $3^3 \cdot 4^2$ (whose symmetry group is not generated by reflections alone, nor by rotations alone [21]).

The symbol $\begin{Bmatrix} m \\ n \end{Bmatrix}$ includes the *cuboctahedron* $\begin{Bmatrix} 3 \\ 4 \end{Bmatrix}$ and the *icosidodecahedron* $\begin{Bmatrix} 3 \\ 5 \end{Bmatrix}$. The *prism* is denoted by $\{\} \times \{n\}$, as being the "rectangular product" of the line-segment $\{\}$ and the n-gon $\{n\}$. In the same notation, the *square*

[21]) See Badoureau's Fig. 67, or Andreini's Fig. 9. It is conceivable that *finite* polytopes of this exceptional kind may occur in higher space; but none has ever been discovered.

could be called $\{\} \times \{\}$, and the *cube* $\{\} \times \{\} \times \{\}$. The symbol $s\begin{Bmatrix}2\\n\end{Bmatrix}$ for the *antiprism* is based on analogy with the *snub cube* [22] $s\begin{Bmatrix}3\\4\end{Bmatrix}$ and the *snub dodecahedron* [22] $s\begin{Bmatrix}3\\5\end{Bmatrix}$.

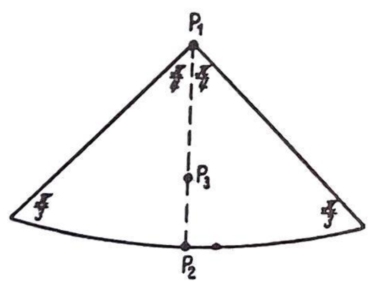
The extended polyhedral group $[3, n]$ ($n = 4$ or 5) is the whole symmetry-group of each of the seven polyhedra

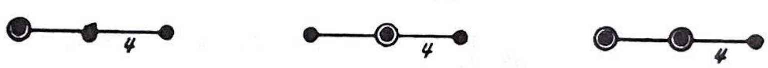
$$\{3, n\}, \quad \begin{Bmatrix}3\\n\end{Bmatrix}, \quad \{n, 3\}, \quad t\{3, n\}, \quad r\begin{Bmatrix}3\\n\end{Bmatrix}, \quad t\{n, 3\}, \quad t\begin{Bmatrix}3\\n\end{Bmatrix},$$

whose graphical symbols are derived from 

by ringing one, two, or all three dots. But the only uniform polyhedra of which $[3, 3]$ is the whole symmetry-group are $\{3, 3\}$ and $t\{3, 3\}$. For, the three polyhedra

have an extra symmetry, which can be observed as a symmetry of the graphical symbols; in fact, the fundamental region of $[3, 3]$ is an isosceles triangle, and the typical vertex (P_1 or P_2 or P_3) lies on the symmetrical median. By adding the reflection in this median we enlarge the group $[3, 3]$ to $[3, 4]$. These three polyhedra are thus the same as





(the octahedron, the cuboctahedron, and the truncated octahedron).

Similarly $[3, n]'$ ($n = 4$ or 5) is the whole symmetry-group of $s\begin{Bmatrix}3\\n\end{Bmatrix}$, but $s\begin{Bmatrix}3\\3\end{Bmatrix}$ is the icosahedron.

§ 1. 6.

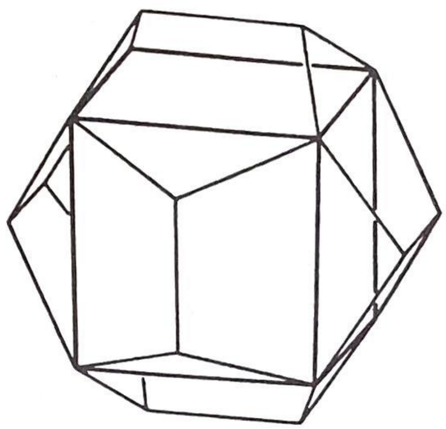
The pyritohedral group.

Kepler [23] observed that the eight vertices of a cube can be selected from the twenty vertices of the dodecahedron, one edge of the cube lying in each of the twelve faces of the dodecahedron. It follows by reciprocation that eight of the twenty faces of the icosahedron lie in the planes of the faces of an

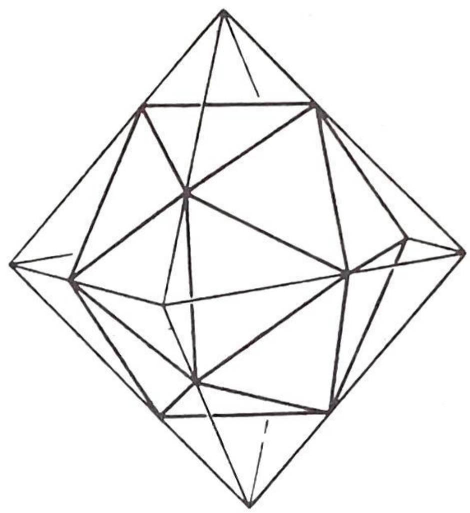
[22]) These, as the symbols indicate, would have been more happily termed "snub cuboctahedron" and "snub icosidodecahedron".

[23]) Kepler. Opera omnia 5, p. 271.

octahedron [24]) whose twelve edges pass respectively through the twelve vertices of the icosahedron.



Cube and dodecahedron.



Icosahedron and octahedron.

The vertices of the octahedron may be supposed to have Cartesian coordinates

$$(\pm 1, 0, 0), \quad (0, \pm 1, 0), \quad (0, 0, \pm 1).$$

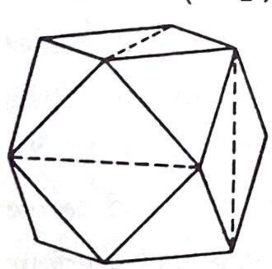
Its edges are divided in the ratio $a : b$ by the points

$$(\pm a, \pm b, 0), \quad (0, \pm a, \pm b), \quad (\pm b, 0, \pm a).$$

These are the vertices of a regular icosahedron [25]) provided the point $(a, b, 0)$ is equidistant from $(a, -b, 0)$ and $(0, a, b)$, that is, provided $\frac{a}{b}$ is the positive root of the equation

$$\tau^2 - \tau - 1 = 0,$$

namely $\frac{a}{b} = \frac{\sqrt{5}+1}{2}$. (The negative root gives analogously the *great icosahedron* $\left\{3, \frac{5}{2}\right\}$, which has the same vertices as a larger $\{3, 5\}$.)



Mrs. A. Boole Stott has pointed out that, if we divide the edges of the octahedron in any ratio (the same for all, so that each face has an inscribed equilateral triangle), we obtain in general an irregular icosahedron with edges of two different lengths. Of the twenty triangular faces, eight are equilateral and twelve isosceles. From this point of view, the cuboctahedron which results from taking the ratio 1 : 1 is to be regarded as a partially degenerate icosahedron, one diagonal of each square being the common base of two isosceles triangles which have come to lie in one plane.

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24) Poinsot, Cauchy, Bertrand, Cayley, Abhandlungen über die regelmäßigen Sternkörper (Leipzig 1906), p. 61.

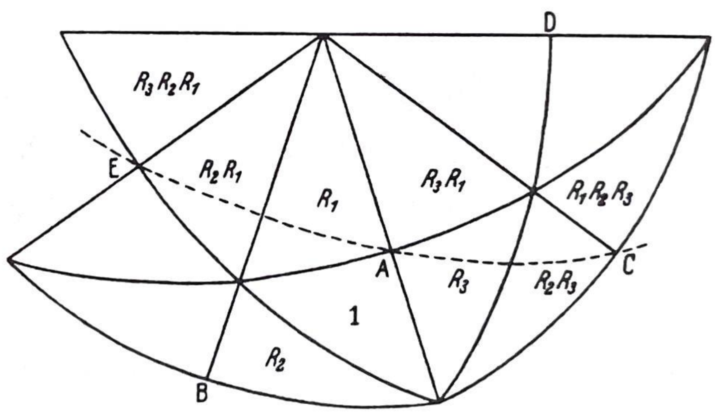
25) P. H. Schoute, Mehrdimensionale Geometrie 2 (Leipzig 1905), p. 158.

The symmetry-group of the figure composed of the octahedron with its inscribed icosahedron, is not [3, 4], since it lacks the reflection which reverses an edge of the octahedron. But it contains the trigonal rotations of the octahedral group (which generate the tetrahedral group) and the reflections that interchange pairs of opposite vertices of the octahedron. It is therefore the pyritohedral group [3′, 4], generated by the operations $R_1 R_2$ and R_3 of [3, 4]. (See (1. 43). Of course this is also symmetry-group of the reciprocal figure composed of the cube with its circumscribed dodecahedron.)

The cube inscribed in the dodecahedron is one of five such cubes (in the same dodecahedron); reciprocally, five octahedra can be circumscribed to a given icosahedron. The permutations of the five cubes (or of the five octahedra) provide a very clear demonstration of the simple isomorphism between the icosahedral group [3, 5]′ and the alternating group of degree five[26]).

§ 1. 7.

The period of $R_1 R_2 R_3$ in $[m, n]$.



The above diagram shows five vertices of the polyhedron $\left\{\begin{matrix} m \\ n \end{matrix}\right\}$ as derived by Wythoff's construction from a network of spherical triangles of angles $\frac{\pi}{2}, \frac{\pi}{m}, \frac{\pi}{n}$. The triangles are named after the corresponding operations of the group $[m, n]$, the first triangle (fundamental region) being named 1. To construct $\left\{\begin{matrix} m \\ n \end{matrix}\right\}$ we take the vertex A at the right angle of triangle 1, and derive the adjacent vertices B, C, D, E by reflecting in the hypotenuses of the four triangles (1, R_3, $R_3 R_1$, R_1) which surround A. Since the whole figure is symmetrical under the rotation through π about A (namely the operation $R_3 R_1$), the points E, A, C lie on a great circle. The operation $R_1 R_2 R_3$, which carries E to A, and A to C, is a rotary reflection, consisting of the rotation from EA to AC combined with the reflection in this great circle.

[26]) F. Klein, Vorlesungen über das Ikosaeder (Leipzig 1884), p. 19.

Suppose that h arcs such as E A or A C make up the whole great circle. Then the period of $R_1 R_2 R_3$ is h or $2h$ according as h is even or odd. (The latter possibility occurs only in the trivial case when $m = 2$ and n is odd.)

Let us now join the five points A, B, C, D, E by straight lines, so that E A B ... is an m-gon, B A C ... is an n-gon, and E A C ... is an h-gon. Then

$$\mathrm{EB} = 2a\cos\frac{\pi}{m}, \quad \mathrm{BC} = 2a\cos\frac{\pi}{n}, \quad \mathrm{EC} = 2a\cos\frac{\pi}{h},$$

where $a = \mathrm{AB}\ (= \mathrm{AC} = \mathrm{AD} = \mathrm{AE})$. But, since BCDE is a rectangle,

$$\mathrm{EC}^2 = \mathrm{EB}^2 + \mathrm{BC}^2.$$

Hence

$$\cos^2\frac{\pi}{h} = \cos^2\frac{\pi}{m} + \cos^2\frac{\pi}{n}. \tag{1.71}$$

By considering the analogous tessellations, we see that this formula continues to hold when $\frac{1}{m} + \frac{1}{n} = \frac{1}{2}$. The results are tabulated below.

m	2	3	3	3	3	4
n	n	3	4	5	6	4
h	n	4	6	10	∞	∞

The equation (1.71) has an application in the theory of regular star-polyhedra [27]. Moreover, if $m > 2$ and $n > 2$ (so that the graph is connected), the total number of reflections is $\frac{3h}{2}$. Comparison with the analogous results in higher space [28] shows that this expression is not purely fortuitous, although no explanation for it has yet been found.

But the most important application of (1.71) is in connection with the central inversion. The operation $(R_1 R_2 R_3)^{\frac{h}{2}}$, which carries the arc A C to its antipodal position, is either the central inversion or a rotation through π, according as it involves an odd or even number of reflections, that is, according as $\frac{h}{2}$ is odd or even. If $m > 2$ and $n > 2$, the reflection in the great circle E A C does not belong to the group; therefore the central inversion belongs *only* when $\frac{h}{2}$ is odd, and is then given by the formula

$$Z = (R_1 R_2 R_3)^{\frac{h}{2}}.$$

[27] Coxeter, Proc. Camb. Phil. Soc. **27** (1931), p. 203.
[28] Coxeter, Annals of Math. **35** (1934), p. 610.

Since the central inversion belongs to $[3, n]$ ($n = 4$ or 5) but not to the rotation group $[3, n]'$, the former group is the direct product of the latter with the group of order two generated by Z; thus we may write

$$[3, n] \sim [1] \times [3, n]' \qquad (n = 4 \text{ or } 5). \tag{1.72}$$

Moreover, Z occurs in the pyritohedral group (see § 1. 4) as $(S_1 R_3)^3$; hence

$$[3', 4] \sim [1] \times [3, 3]'. \tag{1.73}$$

Since $[3, 3]'$ and $[3, 4]'$ are the alternating and symmetric groups of degree four, while $[3, 5]'$ is the alternating group of degree five, it follows that $[3', 4]$ is a subgroup in both $[3, 4]$ and $[3, 5]$. (See § 1. 6).

§ 1. 8.

An explanation for Pappus' observation on reciprocal regular polyhedra.

Pappus noticed that, when two reciprocal regular solids of the same in-radius (and therefore the same circum-radius) stand side by side on a horizontal plane, the manner in which the vertices are distributed in horizontal planes is the same for both: the planes are the same, and the numbers of vertices in each plane are proportional [29]. The situation is similar when we compare the "semi-reciprocal" solids $\{m, n\}$ and $\left\{ {m \atop n} \right\}$, so placed that two opposite edges of the former lie in the same planes as two opposite n-gons of the latter. (Thus the vertices of the dodecahedron, distributed as $2 + 4 + 2 + 4 + 2 + 4 + 2$, lie in the same planes as the vertices of the icosidodecahedron, distributed as $3 + 6 + 3 + 6 + 3 + 6 + 3$.) Wythoff's construction provides a general explanation for these apparently fortuitous results.

Let P, Q be any two points of the sphere on which the group $[m, n]$ operates. Then *the distances from* P *to all the transforms of* Q *are equal (in some order) to the distances from* Q *to all the transforms of* P. For, any operation S of the group transforms the points P, QS^{-1} into the points PS, Q.

Consider two reciprocal solids, $\{m, n\}$ and $\{n, m\}$, derived by Wythoff's construction from the same network of spherical triangles. Let P be any vertex of the former, Q of the latter, so that the transforms of P coincide in sets $2n$ at all the vertices of $\{m, n\}$ while those of Q coincide in sets of $2m$ at the vertices of $\{n, m\}$. When we regard P as lying vertically below the centre, the vertices of $\{n, m\}$ are distributed in horizontal planes according to their distances from P; and when instead we regard Q as lying vertically below the centre, we get the analogous distribution of the vertices of $\{m, n\}$. A similar argument applies to the comparison of $\{m, n\}$ and $\left\{ {m \atop n} \right\}$, the transforms of any one vertex of the latter coinciding in sets of four at all the vertices.

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[29]) Pappus of Alexandria, Book III, Props. 54—58.

The nature of the above proof makes it clear that analogous results will hold in any number of dimensions.

§ 1. 9.

Uniform space-fillings.

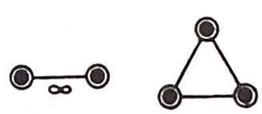
The best way to acquire a clear idea of the appearance of polytopes in four dimensions is to examine first the analogous infinite figures ("solid tessellations") which consist of polyhedral solids fitting together to fill the whole three-dimensional space. Such a space-filling is said to be *uniform* if it has the following two properties:

(i) the solids are uniform polyhedra,

(ii) there is a group of motions (with or without reflections) which is transitive on the vertices.

These figures have been almost exhaustively enumerated by Andreini [30]), who gives excellent stereoscopic photographs of portions of them. Although there are a few anomalous space-fillings [31]) analogous to the plane-filling $3^3 \cdot 4^2$, we shall restrict consideration to those uniform space-fillings which are derivable (from infinite groups generated by reflections) by Wythoff's construction. We also exclude, as trivial, those cases in which the fundamental region is infinite (namely a wedge, or an infinitely tall prism), that is, we exclude those cases in which the graphical symbol falls into two disconnected pieces, of which one represents a finite group and the other an infinite group.

If the graphical symbol is disconnected at all, the fundamental region is a finite prism; and the initial point, which is to be a typical vertex of the space-filling, may be taken either in one of the two basal planes or half-way between them. In the former case we put a ring around one dot in the symbol  , and in the latter case around both dots. The position of the point in its plane is determined by the arrangement of rings in the remaining piece of the disconnected graph. For instance, the symbol



represents the obvious space-filling of hexagonal prisms, the fundamental region being an equilateral triangular prism with the initial point at its centre. (In order that the vertical and horizontal edges of the space-filling may be equal, the vertical and horizontal edges of this triangular prism must be in

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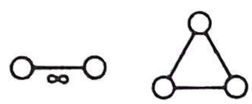
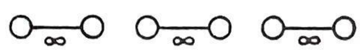
30) Mem. della Soc. ital. delle Sci. (3) **14** (1905), pp. 75–129.

31) Coxeter, Proc. London Math. Soc. (2) **34** (1931), pp. 183–184.

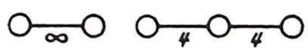
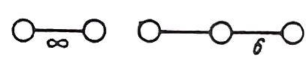
the ratio $1 : \sqrt{3}$). Clearly, the same space-filling is represented if we replace the first piece of the symbol by  (and halve height of the triangular prism), or again if we replace the other piece by  or by . As a "rectangular product" this space-filling is $\{\infty\} \times \{6, 3\}$. Wythoff's construction gives all such products except

$$\{\infty\} \times s\begin{Bmatrix}3\\6\end{Bmatrix}, \quad \{\infty\} \times s\begin{Bmatrix}4\\4\end{Bmatrix}, \quad \{\infty\} \times (3^3.4^2).$$

If we "remove the reflections" and consider only the rotational subgroup of index two, the initial point is determined in the fundamental region by the fact that its distances from the edges are inversely proportional to the sines of the corresponding dihedral angles. Consequently, if the fundamental region is a prism, it must be either an equilateral triangular prism (with vertical and horizontal edges in the ratio $\sqrt{2} : \sqrt{3}$) or a cube; for, the points P and Q of page 393 must coincide. Thus the symbols

(1.91)  and 

represent uniform space-fillings (of tetrahedra and octahedra in both cases, but differently arranged [32]), although

 and 

do not.

If the graphical symbol is connected, the fundamental region is tetrahedral, and we have one of the three groups (1.34). The four dots symbolize the faces of the tetrahedron, or equally well the respectively opposite vertices. Analogy with the two-dimensional case makes it unnecessary to explain in detail what space-fillings are symbolized when we put rings around one, two, three, or all four dots. In order that the edges may be all equal, the initial point has to be equidistant from any two faces whose corresponding dots are ringed; in particular, it has to be at the in-centre of the tetrahedron when all four dots are ringed. Clearly, the symbol for each space-filling contains the symbols for the various solids that occur.

It seems best to postpone till § 2.7 the proof that no uniform space-fillings, apart from those already mentioned, arise by considering rotational subgroups.

We are now ready to symbolize (sometimes in several different ways) all Andreini's uniform space-fillings.

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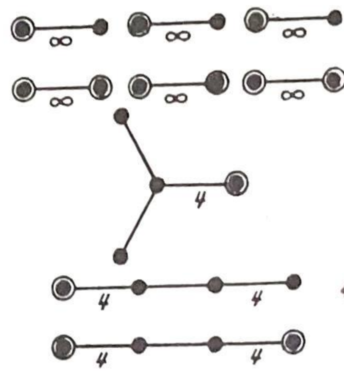
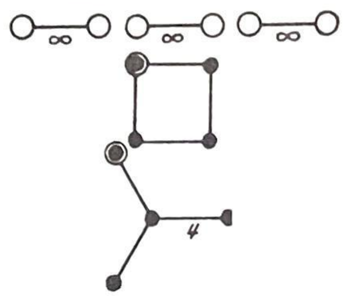
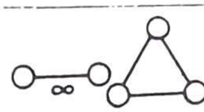
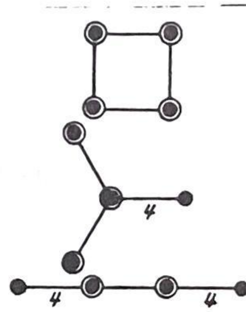
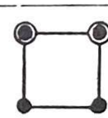
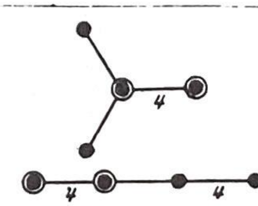
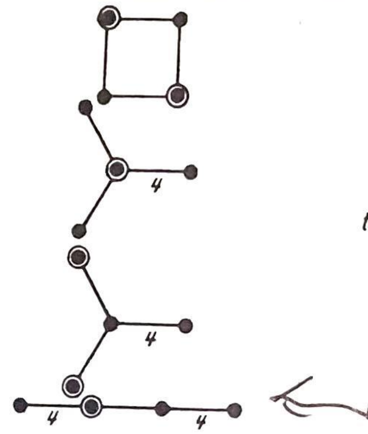
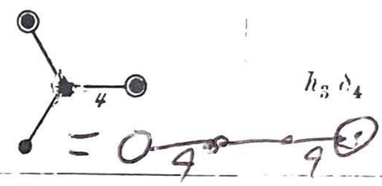
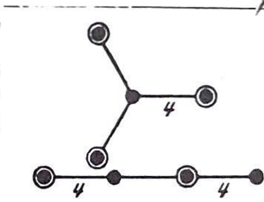
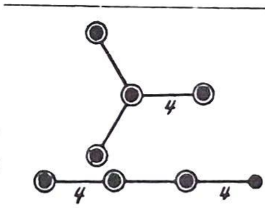
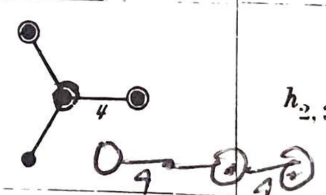
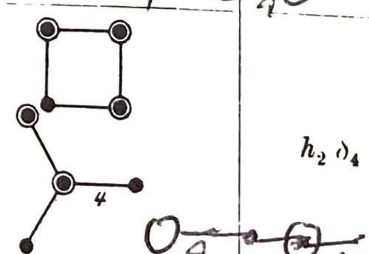
[32]) The vertices are the centres of the spheres in "hexagonal" and "spherical" close-packing, respectively.

Andreini's figure	Solids at each vertex	Graphical Symbol (Alternatives)	Abbreviated symbol [33]
(The ordinary cubic lattice)	8 cubes		δ_4
12	8 tetrahedra, 6 octahedra		$h\,\delta_4$
13	8 tetrahedra, 6 octahedra		
14	4 truncated octahedra		$t_{1,2}\,\delta_4$
15	2 tetrahedra, 6 truncated tetrahedra		$q\,\delta_4$
17[34]	1 octahedron, 4 truncated cubes		$t_{0,1}\,\delta_4$

[33]) Coxeter, Phil. Trans. Royal Soc. London (A) 229 (1930), pp. 344, 365, 360. The symbol δ_4 means the solid tessellation of cubes, regarded as a degenerate four-dimensional polytope; $h\,\delta_4$ and $q\delta_4$ stand for "half δ_4" and "quarter δ_4", since $q\delta_4$ has half the vertices of $h\delta_4$, which in turn has half the vertices of δ_4; $t_{0,1}\,\delta_4$, $t_1\,\delta_4$, $t_{1,2}\,\delta_4$ are "truncations", finite analogues of which will be described in § 2. 4. The remaining symbols are natural generalizations of these.

[34]) Fig. 16 is not uniform.

Andreini's figure	Solids at each vertex	Graphical Symbol (Alternatives)	Abbreviated symbol
18	2 octahedra 4 cuboctahedra		$t_1 \delta_4$
19	1 tetrahedron 1 cube 3 rhombicuboctahedra		$h_3 \delta_4$
20	2 cubes 1 cuboctahedron 2 rhombicuboctahedra		$t_{0,2} \delta_4$
21	1 cube 1 truncated octahedron 2 truncated cuboctahedra		$t_{0,1,2} \delta_4$
22	1 cube 2 octagonal prisms 1 truncated cube 1 rhombicuboctahedron		$t_{0,1,3} \delta_4$
23	1 truncated tetrahedron 1 truncated cube 2 truncated cuboctahedra		$h_{2,3} \delta_4$
24	1 cuboctahedron 2 truncated tetrahedra 2 truncated octahedra		$h_2 \delta_4$
24^{bis}	2 octagonal prisms 2 truncated cuboctahedra		$t_{0,1,2,3} \delta_4$

Whenever a space-filling is derived from the whole group generated by reflections (and not merely from the rotational subgroup), the number of each type of solid at a vertex is easily deduced from the graphical symbol. By removing all the ringed dots, and any links which emanate from them, we derive the symbol for the group which leaves one vertex invariant. (If all the dots are ringed, we are left with the null graph, which represents the group of order one.) By removing all the ringed dots from the symbol for one of the solids, we derive the symbol for the group which leaves invariant both this solid and one of its vertices. The number of such solids at a vertex is clearly equal to the index of the latter subgroup in the former.

Consider, for example, the symbol

Here the group which keeps one vertex fixed is of order four, being represented by two isolated dots. The solids are truncated octahedra of the two "types":

The group which keeps fixed both a solid and one of its vertices, is of order two, being represented by a single dot. The index of the latter group in the former is two, showing that the vertex is surrounded by two truncated octahedra of either type, that is, by four altogether.

Clearly, [4, 3, 4] is the whole symmetry-group of δ_4, $t_{0,1}\,\delta_4$, $t_1\,\delta_4$, $t_{0,2}\,\delta_4$, $t_{0,1,2}\,\delta_4$ and $t_{0,1,3}\,\delta_4$; [[4, 3, 4]], of $t_{1,2}\,\delta_4$ and $t_{0,1,2,3}\,\delta_4$; $\begin{bmatrix}3\\3\\4\end{bmatrix}$, of $h\,\delta_4$, $h_3\,\delta_4$, $h_{2,3}\,\delta_4$, $h_2\,\delta_4$; and [□], of $q\,\delta_4$.

Appendix:
On Cayley group-pictures.

The idea of representing the operations of any group by the points of a graph has been proposed by both Cayley and Dehn [35]). Every line of the graph is directed and is made to correspond to a definite generator of the group. (Cayley does this by colouring the lines, with different colours for the different generators.) At each point there are two lines for each generator, one directed towards the point and one directed away. Given any operation of the group, expressed in terms of the generators, we may start out from any

[35]) See W. Burnside, Theory of Groups (Cambridge 1911), pp. 423—427; W. Threlfall, Gruppenbilder (Leipzig 1932), pp. 22—27; W. Gruner, Comm. Math. Helv. **10** (1938), p. 53.

one point of the graph and proceed along a representative path to some other (definite) point. Any different path between the same two points will represent some other expression for the same operation. Hence each relation in the abstract definition for the group will be represented by a closed path.

Whenever we have an involutory generator, the corresponding pairs of lines may be replaced by single lines along which both directions are allowed. For instance, a hexagon of such undirected lines represents the dihedral group [3], defined by

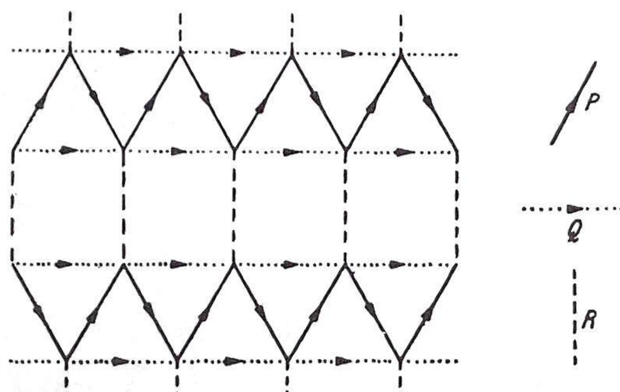
$$R_1^2 = R_2^2 = (R_1 R_2)^3 = 1.$$

On the other hand, a hexagon of cyclically directed lines represents the cyclic group [6]′, defined by

$$S^6 = 1.$$

In many cases the graph may be taken to consist of the vertices and edges of a polyhedron. Burnside's frontispiece shows the snub cube, $s\begin{Bmatrix}3\\4\end{Bmatrix}$, as a group-picture for the octahedral group [3, 4]′. The polyhedron being simply-connected, its different faces (together with any edge along which both directions are allowed) provide the generating relations for an abstract definition. For instance, the anomalous tessellation $3^3 \cdot 4^2$ represents the infinite group

$$P^2 = Q, \quad R^2 = 1, \quad QR = RQ.$$



The following table shows that *every uniform polyhedron and tessellation, except the dodecahedron* $\{5, 3\}$ *and the icosidodecahedron* $\begin{Bmatrix}3\\5\end{Bmatrix}$, *can be regarded as a Cayley group-picture* (sometimes of two distinct groups, by directing the edges differently)[36]).

36) This fact was partially recognized by Maschke, Amer. Journ. of Math. 18 (1896), pp. 156—194. His Figs. 2–10, 16–18 are respectively $\{\ \}\times\{n\}$, $t\{3, 3\}$, $r\begin{Bmatrix}3\\3\end{Bmatrix}$, $t\{3, 4\}$, $t\{4, 3\}$, $r\begin{Bmatrix}3\\4\end{Bmatrix}$, $t\{3, 5\}$, $t\{5, 3\}$, $r\begin{Bmatrix}3\\5\end{Bmatrix}$, $\{\ \}\times\{2n\}$, $t\begin{Bmatrix}3\\3\end{Bmatrix}$, $t\begin{Bmatrix}3\\4\end{Bmatrix}$.

Group	Abstract definition	Cayley group-picture (See page 394)
$[m, n]$	(1.41)	$t\begin{Bmatrix} m \\ n \end{Bmatrix}$
$[1] \times [n]$	(The same with $m = 2$)	$\{\} \times \{2n\}$
$[m, n]'$	$S_1^m = S_2^n = (S_1 S_2)^2 = 1$	$r\begin{Bmatrix} m \\ n \end{Bmatrix}$
$[n]$	(The same with $m = 2$)	$\{\} \times \{n\}$
$[m, n]'$	$S_2^n = S_0^2 = (S_2 S_0)^m = 1$	$t\{m, n\}$
$[m, n]'$	$S_0^2 = S_1^m = S_2^n = S_0 S_1 S_2 = 1$	$s\begin{Bmatrix} m \\ n \end{Bmatrix}$
$[n]$	(The same with $m = 2$)	$s\begin{Bmatrix} 2 \\ n \end{Bmatrix}$
$[m', 2p]$	$S^m = R^2 = (S^{-1} R S R)^p = 1$	$t\{2p, m\}$
Δ	(1.42)	$\{6, 3\}$
Δ'	$S_1^3 = S_2^3 = (S_1 S_2)^3 = 1$	$\begin{Bmatrix} 3 \\ 6 \end{Bmatrix}$
Δ'	$S_0^3 = S_1^3 = S_2^3 = S_0 S_1 S_2 = 1$	$\{3, 6\}$

The cube is included as $\{\} \times \{4\}$, the tetrahedron as $s\begin{Bmatrix} 2 \\ 2 \end{Bmatrix}$, the octahedron as $s\begin{Bmatrix} 2 \\ 3 \end{Bmatrix}$, the icosahedron as $s\begin{Bmatrix} 3 \\ 3 \end{Bmatrix}$, the cuboctahedron as $r\begin{Bmatrix} 3 \\ 3 \end{Bmatrix}$, and the square tessellation as $r\begin{Bmatrix} 4 \\ 4 \end{Bmatrix}$.

The first two lines of the above table illustrate the following general principle. The Cayley group-picture for any group generated by reflections naturally has one vertex in each region (§ 1. 2), so we can think of it as consisting of the vertices and edges of the polytope symbolized by ringing all the dots of the graph (§ 1. 5). In other words, the network of regions and the Cayley group-picture are dual complexes. The generators being involutory, there is no question of directing the edges of the polytope: both directions are allowed along every edge. Thus, in the notation of § 1. 9, the space-fillings δ_4, $t_{1,2}\,\delta_4$, $t_{0,1,2}\,\delta_4$, $t_{0,1,2,3}\,\delta_4$ provide Cayley group-pictures for $[\infty]\times[\infty]\times[\infty]$, $\square$, $\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$, $[4, 3, 4]$, respectively.

The next three lines of the table illustrate the following less obvious principle. *The rotational subgroup of the group respresented by any connected graph has a Cayley group-picture consisting of the vertices and edges of the polytope symbolized by ringing all but one of the dots of the graph.* (We obtain various group-pictures for the same group by varying the unringed dot.) Let us prove this for the case when the graph has four dots, so that the fundamental region is a tetrahedron bounded by planes p_1, p_2, p_3, p_4. If the first three dots are ringed, the typical vertex of the polytope lies in the face p_4

of the tetrahedron, and the neighbouring vertices are derived by rotating about the sides of this face, i. e. by applying the operations $R_1 R_4$. $R_4 R_1$; $R_2 R_4$, $R_4 R_2$; $R_3 R_4$, $R_4 R_3$. Of the edges leading to these neighbouring vertices, let the first of each pair be directed away from the original vertex, and the second towards it. A consistent direction for all the edges of the polytope is then given by applying the rotation-group; and the vertices and edges of the polytope form a Cayley group-picture for this group, as generated by $R_1 R_4$, $R_2 R_4$, $R_3 R_4$ [37]). Whenever one of the dihedral angles of the tetrahedron is a right angle, the corresponding pair of edges of the polytope will coincide, in the manner appropriate for an involutory generator.

The examples in ordinary space are given in the first six lines of the following table. The plane interfaces of each space-filling provide the relations for the abstract definition of the group. This last remark enables us to insert the remaining lines of the table. The abstract definitions can be verified by writing $A = R_1 R_4$, $B = R_2 R_4$, $C = R_3 R_4$, $D = R_1 R_3$, $E = R_2 R_3$, $F = R_1 R_2$, $G = R_4$. $H = FA$, $J = FD$, $K = F^2$, $L = C^2$, in the notation of (1. 45).

	Group	Abstract definition	Cayley group-picture
$[4,3,4]^+$	$\{4, 3, 4\}'$	$A^2 = B^2 = C^4 = (AB)^4 = (AC)^2 = (BC)^3 = 1$	$t_{0,1,2}\, \delta_4$
		$C^4 = D^2 = E^2 = (CD)^2 = (CE)^2 = (DE)^4 = 1$	$t_{0,1,3}\, \delta_4$
	$\begin{bmatrix}3\\3\\4\end{bmatrix}'$	$H^2 = B^2 = C^4 = (HB)^2 = (HC)^3 = (BC)^3 = 1$	$t_{1,2}\, \delta_4$
$[4,3^{1,1}]^+$		$C^4 = J^3 = E^3 = (CJ)^2 = (CE)^2 = (JE^{-1})^2 = 1$ [38])	$t_{0,2}\, \delta_4$
		$K^2 = B^2 = E^3 = (KB)^2 = (KE)^3 = (BE)^4 = 1$	$h_{2,3}\, \delta_4$
$[3^{[4]}]^+$	□′	$L^2 = J^3 = E^3 = (LJ)^3 = (LE)^3 = (JE^{-1})^2 = 1$ [39])	$h_2\, \delta_4$
$[(4,3)^+,4]$	$[(4.3)', 4]$	$E^3 = F^4 = G^2 = (EF)^2 = (E^{-1}GEG)^2 = F^{-1}GFG = 1$	$t_{0,1,3}\, \delta_4$
$[(4,3,4,2^+)]$	$(4, 4 \mid 3, 3)$	$R^4 = S^4 = (RS)^3 = (R^{-1}S)^3 = 1$	$t_{1,2}\, \delta_4$
$[[3^{[4]}]^+]$	$(6, 6 \mid 2, 3)$	$U^6 = V^6 = (UV)^2 = (U^{-1}V)^3 = 1$	$t_{1,2}\, d_4$
$[[4,3,4]^+]$	$(4, 6 \mid 2, 4)$	$X^4 = Y^6 = (XY)^2 = (X^{-1}Y)^4 = 1$	$t_{0,1,2,3}\, \delta_4$

37) The abstract definition is

$$S_1^{n_{14}} = S_2^{n_{24}} = S_3^{n_{34}} = (S_2^{-1} S_3)^{n_{23}} = (S_3^{-1} S_1)^{n_{31}} = (S_1^{-1} S_2)^{n_{12}} = 1.$$

38) The same relations, with JE instead of JE^{-1}, would define a group of order 192.

39) Here JE could be written for JE^{-1}, by defining E as $R_3 R_2$ instead of $R_2 R_3$

(Eingegangen am 8. Mai 1939.)