5

REGULAR SKEW POLYHEDRA IN THREE AND FOUR DIMENSIONS, AND THEIR TOPOLOGICAL ANALOGUES

Reprinted, with the editors' permission,
from the *Proceedings of the London Mathematical Society*,
Ser. 2, Vol. 43, 1937.

The five regular ("Platonic") polyhedra,

$$\{3, 3\}, \quad \{3, 4\}, \quad \{4, 3\}, \quad \{3, 5\} \quad \{5, 3\}^*,$$

have been known and admired for thousands of years. As analogues of these, Kepler drew attention to the three regular plane-fillings,

$$\{4, 4\}, \quad \{3, 6\}, \quad \{6, 3\},$$

which may be regarded as regular polyhedra with infinitely many faces. He also introduced two of the four *star* polyhedra,

$$\{5, \tfrac{5}{2}\}, \quad \{\tfrac{5}{2}, 5\}, \quad \{3, \tfrac{5}{2}\}, \quad \{\tfrac{5}{2}, 3\}†;$$

the remaining two were added by Poinsot, but it was Cauchy‡ who first proved that there are *only* four. These have the disadvantage of possessing "false vertices", *i.e.* points other than vertices, where three or more faces meet.

One day in 1926, J. F. Petrie told me with much excitement that he had discovered two new regular polyhedra; infinite, but free from false vertices. When my incredulity had begun to subside, he described them to me: one consisting of squares, six at each vertex (Fig. i), and one

---

* The meaning of these symbols, introduced by Schläfli, is clear from the following example: the *cube* is {4, 3}, because it is bounded by *squares*, *three* at each vertex.

† This extension of the above notation is also due to Schläfli. The four polyhedra are, respectively, the *great dodecahedron*, the *small stellated dodecahedron*, the *great icosahedron*. the *great stellated dodecahedron*.

‡ Cauchy, 1. *Cf.* Coxeter, 1, 203. **(For references, see page 105.)**

consisting of hexagons, four at each vertex (Fig. ii). It was useless to protest that there is *no room* for more than four squares round a vertex. The trick is, to let the faces go up and down in a kind of zig-zag formation, so that the faces that adjoin a given "horizontal" face lie alternately "above" and "below" it. When I understood this, I pointed out a third possibility: hexagons, six at each vertex (Fig. iii).

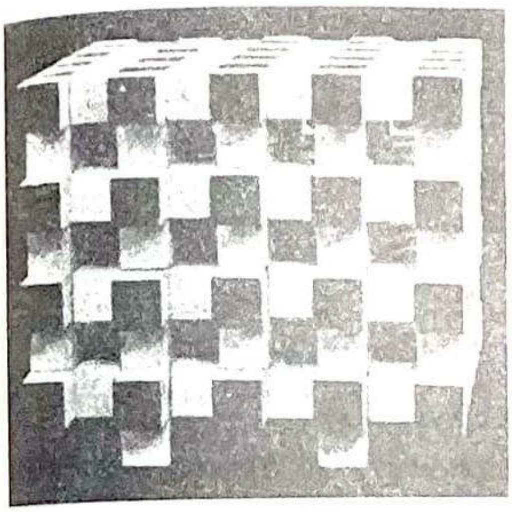


Fig. i: {4, 6|4}.

Fig. ii: {6, 4|4}.

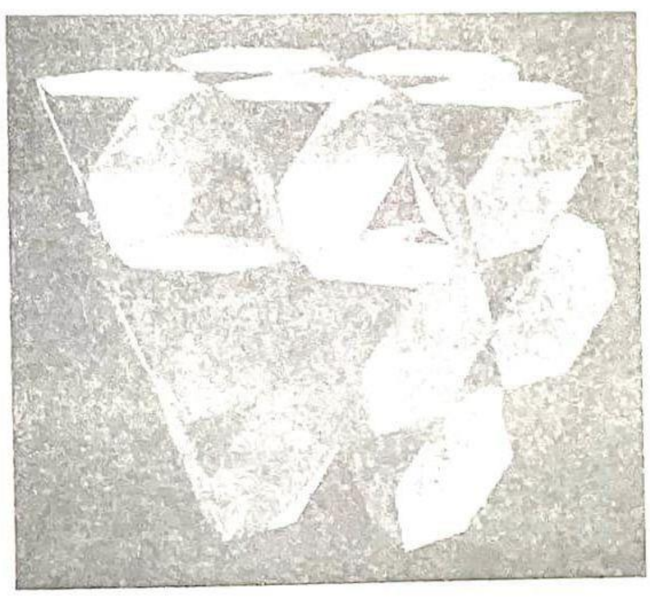


Fig. iii: {6, 6|3}.

It then occurred to us that, although these new polyhedra are infinite, we might find analogous *finite* polyhedra by going into four-dimensional space. Petrie cited one consisting of n^2 squares, four at each vertex. We call such figures *skew polyhedra*, by analogy with "skew polygons". They must not be confused with *polytopes* (which have solid faces). Actually, every regular skew polyhedron is intimately associated with a definite *uniform** *polytope*. *E.g.*, Fig. i consists of half the squares of $\{4, 3, 4\}$ or δ_4, the net of cubes; Fig. ii consists of all the hexagons of $t_{1,2}\delta_4$, the net of truncated octahedra; and Fig. iii† consists of all the hexagons of $q\delta_4$, the net of tetrahedra and truncated tetrahedra‡.

The reader naturally asks, What do we mean by saying that these new polyhedra are *regular*? To explain this, we have to go back to first principles.

1. *Definitions*.

We define a *polyhedron* as a connected set of ordinary plane polygons‖, such that every side of every polygon belongs also to just one other polygon of the set. We stipulate that any two of the polygons shall have in common either a side (and two vertices), or a single vertex, or nothing at all (thus ruling out the Kepler-Poinsot polyhedra). We define a *symmetry* (or "symmetry operation") of any figure as a congruent transformation of the figure into itself (*i.e.*, a combination of translations, rotations, and reflections). A completely irregular figure has no symmetry save identity.

A polyhedron is said to be *regular* if it possesses two particular symmetries: one which cyclically permutes the vertices of any face c, and one which cyclically permutes the faces that meet at a vertex C, C being a vertex of c. It follows that these two symmetries, say R and S, generate a group which is transitive on the vertices, on the edges, and on the faces. Moreover, the faces are regular polygons (in the most elementary sense). This clearly agrees with our preconceived notion of regularity.

Perhaps this will seem clearer if we give the analogous definition for a regular *polygon*. A polygon (which may be skew) is said to be regular if it possesses a symmetry which cyclically permutes the vertices (and therefore also the sides) of the polygon. Regular polygons can be classified

---

* Coxeter, 4.

† For this picture some apology is required. The lines which divide each hexagon into six triangles are to be ignored, since they arise merely from the manner in which this particular model was built up.

‡ Andreini, 1, 106 (Fig. 14), 107 (Fig. 15).

‖ In § 5, we relax this definition, and consider a connected set of *topological* polygons.

according to the nature of this symmetry. The cases that arise in ordinary space are as follows:

Symmetry.	*Polygon.*
Reflection	Digon, {2}, with two vertices and two coincident sides.
Rotation (through $2\pi/n$)	Ordinary n-gon, which we denote by $\{n\}$.
Rotary reflection (involving rotation through π/n)	Finite skew $2n$-gon.
Translation	Apeirogon, $\{\infty\}$ (*i.e.*, an infinite straight line broken into equal segments).
Glide	Plane zig-zag (dividing the plane into two equal parts).
Screw	Helical polygon.

Since the "square" of a rotary reflection is a pure rotation, the vertices of a regular *finite* skew polygon lie alternately on two circles which reflect into one another in a plane parallel to the planes of the circles (and therefore half-way between them). In fact, the sides of such a polygon are the lateral edges of an *antiprism* (*e.g.*, Fig. iv). Hence the number of sides must be *even*. (If there are only four sides, these are edges of a tetragonal bisphenoid, which can be regarded as a *digonal* antiprism.)

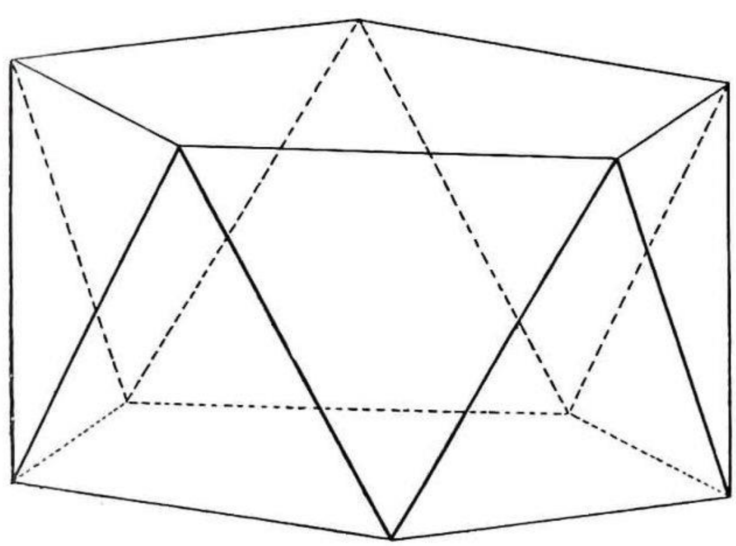


Fig. iv: The skew decagon in a pentagonal antiprism.

The line joining two alternate vertices of an ordinary regular n-gon (of unit side) is of length $2\cos(\pi/n)$, and is called the *vertex figure* of the $\{n\}$.

For a regular polyhedron (of unit edge), consider those vertices which are joined by edges to any one vertex. These can be regarded as the vertices of a finite polygon whose sides are vertex figures of faces of the polyhedron. This polygon is called the *vertex figure* of the polyhedron. It is a *regular* polygon, since its sides are cyclically permuted by the special symmetry S.

If a regular polyhedron is four-dimensional and finite, its vertices are equidistant from their centroid, which we call the *centre* of the polyhedron. Hence those vertices which are joined by edges to any one vertex lie on the intersection of two hyper-spheres, *i.e.* on an ordinary sphere. Thus in this case, as well as when the polyhedron (finite or infinite) is three-dimensional, the vertex figure lies in a three-space. By taking as vertex figure a plane polygon, we obtain an ordinary regular solid or plane-filling; but the vertex figure may just as well be a skew polygon, whose vertices lie alternately in two parallel planes.

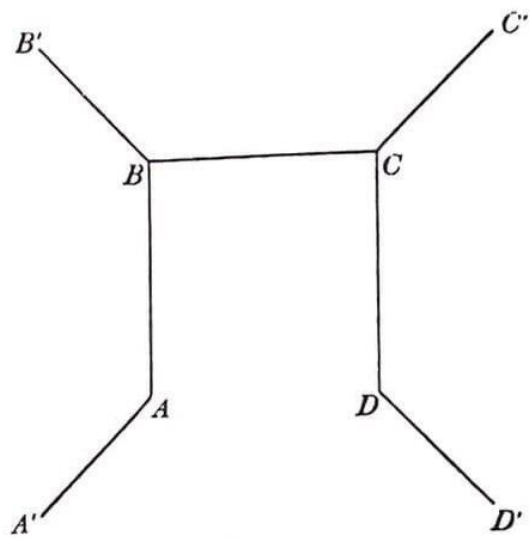


Fig. v.

Let us first suppose the polyhedron to lie in ordinary space. It divides space into two parts, one "inside" and one "outside" the polyhedron. When the vertex figure is a skew polygon, the inside and outside are alike (*i.e.* congruent) in the neighbourhood of one vertex, and are therefore alike altogether. (In fact, R and S each interchange the inside and outside.) Hence the polyhedron must be infinite. On account of the zig-zag nature of the skew polygon, the faces that surround a given "horizontal" face are alternately "above" and "below" it. Hence R, like S, is a rotary reflection.

The situation when the polyhedron is finite in four dimensions is very similar. For, after projecting the polyhedron on to its circumscribed hyper-sphere, we have again a surface lying in a manifold of three dimensions, and dividing that manifold into two equal parts. R and S are still rotary reflections, when regarded as operating in the spherical three-space.

Let A′ABB′..., B′BCC′..., C′CDD′... (Fig. v) be three faces of any regular polyhedron. We may suppose that R cyclically permutes the edges B′B, BC, CC′, ... which belong to the second of these faces, while S cyclically permutes the edges CD, CC′, CB, ... which meet at C. Then the product RS reverses the edge BC, while the "quotient" RS^{-1} cyclically permutes the edges AB, BC, CD, ... Hence ABCD... is a regular polygon (but may be skew). The lines AA′, BB′, CC′, DD′, ... are clearly concurrent, say at O. (When the faces are squares, so that these edges are parallel, we say that O is at infinity.) If the polyhedron is in three dimensions, the planes ABB′, DCC′ are equally inclined to the plane B′BCC′, on the same side of it; therefore ABCD ... is a plane polygon. If the polyhedron is finite in four dimensions, the symmetry RS^{-1}, being the product of two rotary reflections, is a *positive* transformation. Since it leaves the centre invariant, this must be either a simple rotation or a double rotation. Since it also leaves the point O invariant, it can only be a simple rotation. Hence ABCD... is again a plane polygon.

In Figs. i, ii, iii, this polygon appears as a hole (triangular in Fig. iii, square in Figs. i and ii), and it is convenient to adopt the term *hole* in other cases, too. The hole may be described as a path along edges, such that at the end of each edge we leave two faces on (say) the left. In other words, at any vertex of the path, the edges to be selected are not adjacent but alternate. When (as above) the hole is a plane polygon, we may determine it from its vertex figure, which is the join of two alternate vertices of the vertex figure of the polyhedron. In the case of the ordinary polyhedron $\{l, m\}$, whose vertex figure is an $\{m\}$ of side $2\cos(\pi/l)$, it follows that the "hole"* is an $\{n\}$, where

$$2\cos(\pi/n) = 2\cos(\pi/l) \,.\, 2\cos(\pi/m),$$

i.e.

$$\cos(\pi/n) = 2\cos(\pi/l)\cos(\pi/m). \tag{1.1}$$

We shall use the symbol $\{l, m \mid n\}$ to denote any regular polyhedron which is uniquely determined by

$$\begin{cases} l, \text{ the number of vertices or sides of a face,} \\ m, \text{ the number of edges or faces at a vertex,} \\ n, \text{ the number of vertices or sides of a hole.} \end{cases}$$

* Here the term "hole" is not entirely appropriate. But triangular "pits" are clearly visible in the star polyhedron $\{5, \frac{5}{2}\}$ (Fig. xiii).

We shall see that every regular polyhedron in three dimensions, or finite in four, is so determined. From this point of view,

$$\{n,\ 3\} \quad \text{is} \quad \{n,\ 3\,|\,n\},$$
$$\{3,\ n\} \quad \text{is} \quad \{3,\ n\,|\,n\},$$
$$\{4,\ 4\} \quad \text{is} \quad \{4,\ 4\,|\,\infty\}.$$

2. *The trigonometrical criterion for* $\{l,\ m\,|\,n\}$.

Consider first the general infinite regular polyhedron in three dimensions. Let λ, λ' denote the planes that perpendicularly bisect two adjacent edges, BC, CC′, belonging to the face B′BCC′ ... (Fig. v), and let μ, μ' denote the planes that bisect the dihedral angles at those same edges. These four planes bound a tetrahedron, whose angles we proceed to calculate.

The planes λ, μ, and likewise λ', μ', are obviously perpendicular. The planes λ, λ' are both perpendicular to the plane BCC′, and meet it in the perpendicular bisectors of two adjacent sides of the $\{l\}$ B′BCC′ ...; so the angle between them is $2\pi/l$. The bisecting planes of the dihedral angles at the edges through C all pass through the axis of the rotary reflection S; since there are m such edges, the angle between two consecutive bisecting planes, such as μ, μ', is $2\pi/m$. The planes λ, μ' are both perpendicular to the plane BCD, and meet it in the bisectors of the side BC and angle BCD of the $\{n\}$ ABCD ...; so the angle between them is π/n. Collecting results, we see that the six dihedral angles of the tetrahedron $\lambda\lambda'\mu\mu'$ are

$$(2.1)\qquad \begin{cases} (\lambda\mu) = (\lambda'\mu') = \tfrac{1}{2}\pi, \\ \qquad\qquad (\lambda\lambda') \ = 2\pi/l, \\ \qquad\qquad (\mu\mu') = 2\pi/m, \\ (\lambda\mu') = (\lambda'\mu) \ = \pi/n. \end{cases}$$

We have now reduced the problem of enumerating the possible regular polyhedra to that of enumerating the possible tetrahedra of a special type. For each tetrahedron, we have *two* polyhedra,

$$\{l,\ m\,|\,n\} \quad \text{and} \quad \{m,\ l\,|\,n\},$$

which may be called *reciprocal*, since the vertices of each are the centres of the faces of the other. Thus $\{4,\ 6\,|\,4\}$ and $\{6,\ 4\,|\,4\}$ (Figs. i and ii) are reciprocal, while $\{6,\ 6\,|\,3\}$ (Fig. iii) is self-reciprocal.

Now, just as the three angles of a plane triangle are not independent, but add up to π, so also the six dihedral angles of a tetrahedron are not independent, but satisfy the relation

$$(2.2)\quad \Delta \equiv \begin{vmatrix} 1 & -\cos(ab) & -\cos(ac) & -\cos(ad) \\ -\cos(ba) & 1 & -\cos(bc) & -\cos(bd) \\ -\cos(ca) & -\cos(cb) & 1 & -\cos(cd) \\ -\cos(da) & -\cos(db) & -\cos(dc) & 1 \end{vmatrix} = 0,$$

where a, b, c, d are the four bounding planes. In the case under consideration,

$$\Delta = \left(4\cos^2\frac{\pi}{l}\cos^2\frac{\pi}{m} - \cos^2\frac{\pi}{n}\right)\left(4\sin^2\frac{\pi}{l}\sin^2\frac{\pi}{m} - \cos^2\frac{\pi}{n}\right).$$

After discarding two definitely positive factors, we are left with the condition

$$(2.3)\quad \left(2\cos\frac{\pi}{l}\cos\frac{\pi}{m} - \cos\frac{\pi}{n}\right)\left(2\sin\frac{\pi}{l}\sin\frac{\pi}{m} - \cos\frac{\pi}{n}\right) = 0.$$

By (1.1), the vanishing of the first factor gives the ordinary polyhedron $\{l, m\}$. Hence, to find all the possible skew polyhedra (in three dimensions), we merely have to solve in integers the equation

$$(2.4)\quad 2\sin(\pi/l)\sin(\pi/m) = \cos(\pi/n).$$

Apart from the plane-fillings

$$\{3, 6|6\},\quad \{6, 3|6\},\quad \{4, 4|\infty\},$$

which are already covered by (1.1)*, the only solutions are:

$$\{4, 6|4\},\quad \{6, 4|4\},\quad \{6, 6|3\}.$$

These infinite polyhedra may be called "complements" of the cube, octahedron, and tetrahedron, respectively, since each solution $\{l, m|n\}$ of (2.4) corresponds to a solution $\{l', m'|n\}$ of (1.1), where

$$1/l + 1/l' = 1/m + 1/m' = \tfrac{1}{2}.$$

We shall see later that "complementary" polyhedra have the same dihedral angles.

For each value of n, the relation between l and m may be presented graphically by taking l and m as Cartesian coordinates in a plane. (2.3) is then the equation for a couple of curves, which are shown in Fig. vi when $n = 3$, Fig. vii when $n = 4$. The finite polyhedra $\{l, m\}$ appear as O's on the curve

$$\cos(\pi/l)\cos(\pi/m) = \tfrac{1}{2}\cos(\pi/n),$$

* For the plane-fillings we have $1/l + 1/m = \frac{1}{2}$, so they can be regarded indifferently as ordinary or skew polyhedra.

while the infinite polyhedra $\{l, m \mid n\}$ appear as dots on the curve

$$\sin(\pi/l)\sin(\pi/m) = \tfrac{1}{2}\cos(\pi/n).$$

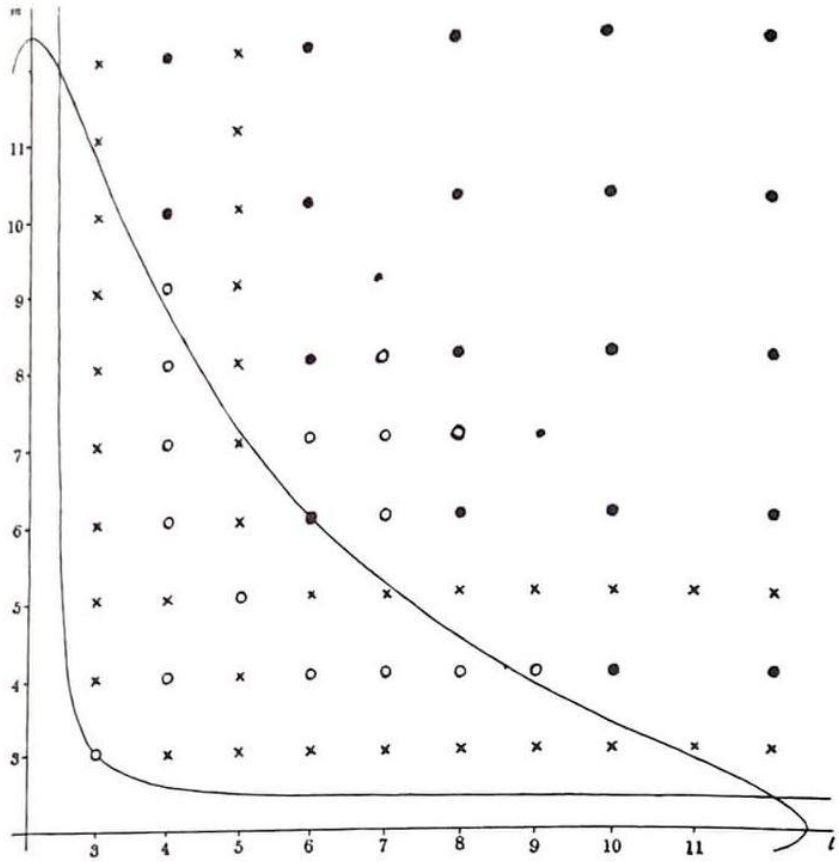


Fig. vi: Graphical enumeration of polyhedra $\{l, m \mid 3\}$.

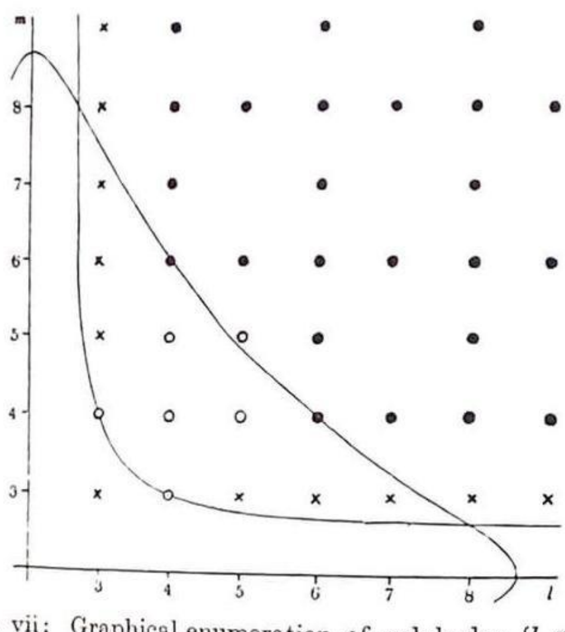


Fig. vii: Graphical enumeration of polyhedra $\{l, m \mid 4\}$.

In the case of the general finite regular polyhedron in four dimensions, the planes $\lambda, \lambda', \mu, \mu'$ have to be replaced by three-spaces through the centre, but the angles between them are still given by (2. 1). They can be regarded as bounding a spherical tetrahedron on the circumscribed hyper-sphere of the polyhedron. As before, the two polyhedra $\{l, m|n\}$ and $\{m, l|n\}$ are derivable from the same tetrahedron, and are *reciprocal* in the sense that the vertices, edges, and faces of one correspond to the faces, edges, and vertices of the other. The equation (2.2) has to be replaced by the inequality*

$$\Delta > 0,$$

which reduces to

$$\left(2 \cos \frac{\pi}{l} \cos \frac{\pi}{m} - \cos \frac{\pi}{n}\right)\left(2 \sin \frac{\pi}{l} \sin \frac{\pi}{m} - \cos \frac{\pi}{n}\right) > 0.$$

It is easily seen that the only significant case is when these two factors are positive†. *E.g.*, in Figs. vi and vii, the significant lattice points are those which lie inside the region enclosed by the two curves.

Since we restrict consideration to finite polyhedra in four dimensions, R and S (as we have seen) are rotary reflections; hence their periods, l and m, must be *even*. Thus the only admissible solutions are:

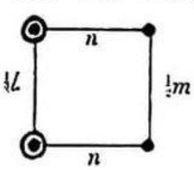
$$\{4, 4|n\},$$

$$\{4, 6|3\}, \quad \{6, 4|3\},$$

$$\{4, 8|3\}, \quad \{8, 4|3\}.$$

3. *Derivation of the polyhedra from uniform polytopes.*

Clearly, the planes or three-spaces $\lambda, \lambda', \mu, \mu'$ are primes of symmetry of the polyhedron, and the reflections in them generate a sub-group of the complete symmetry-group of the polyhedron. (R and S generate a different subgroup.) The vertex C (Fig. v) is the mid-point of the edge $\mu\mu'$ of the tetrahedron $\lambda\lambda'\mu\mu'$, and lies on the bisector of the opposite dihedral angle $(\lambda\lambda')$. The reflections in λ, λ' transform C into B, C′, respectively, while the reflection in μ' transforms B into D. In fact, Wythoff's construction‡ leads to the uniform polytope



whose vertices, edges,

---

* Coxeter, 2, 137.

† When the second factor is negative (and the first positive), the space is Minkowskian instead of Euclidean.

‡ Coxeter, 4, 329.

$\{l\}$'s, and $\{n\}$'s are the vertices, edges, faces, and holes of $\{l, m \mid n\}$. Moreover, the vertex figure of the polytope is an antiprism, whose lateral edges form the vertex figure of the polyhedron.

When $l = m = 4$, the polytope is the double n-gonal prism, $[\{n\}, \{n\}]$ or $\{n\} \times \{n\}$, bounded by $2n$ n-gonal prisms. The corresponding polyhedron,

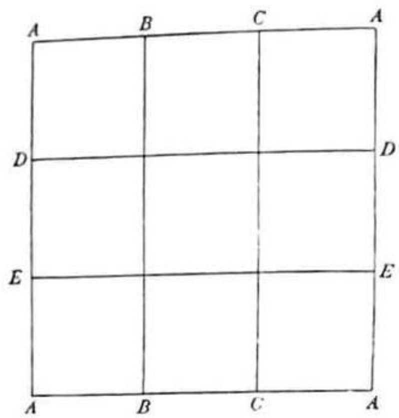


Fig. viii: "Net" of $\{4, 4 \mid 3\}$.

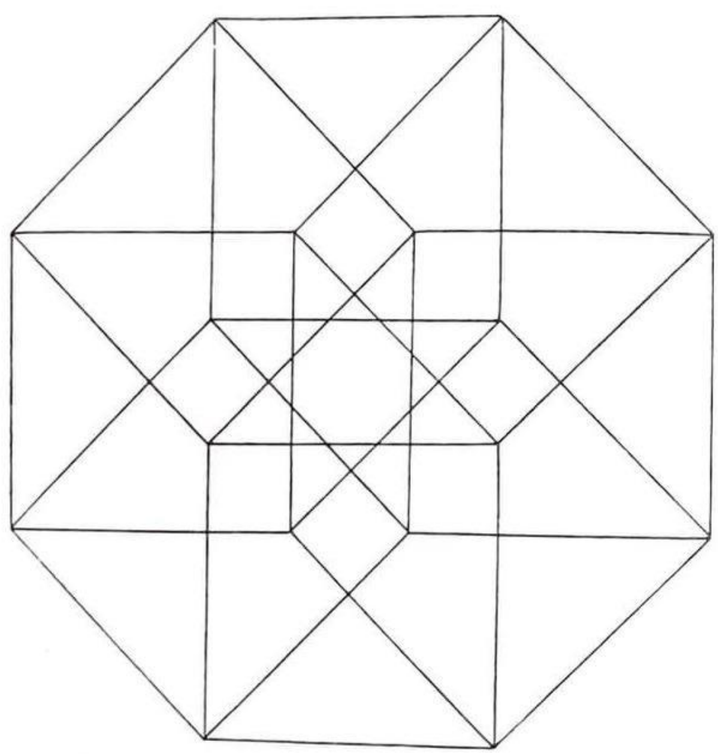


Fig. ix: Orthogonal projection of $\{4, 4 \mid 4\}$.

$\{4, 4 \mid n\}$, can be unfolded into a "net" which consists of n^2 squares forming a large square. The opposite sides of this large square are to be identified (or brought into coincidence) in the manner appropriate to the formation of a torus. (See, for instance, Fig. viii, where $n = 3$. The "squaredpaper" plane-filling, $\{4, 4\}$, can be regarded as the limiting form as n tends to infinity.) When $n=4$, the double prism becomes the tessaract or hyper-cube, γ_4, and the faces of $\{4, 4|4\}$ are sixteen of the twenty-four squares of this polytope. The vertex figure of γ_4 is a regular tetrahedron (of edge $\sqrt{2}$), which can be regarded as a digonal antiprism in three ways. Therefore one γ_4 leads to three $\{4, 4|4\}$'s. Fig. ix shows the familiar octagonal projection of the tessaract (in which the eight bounding cubes are all plainly visible). In this projection, eight of the twenty-four squares are undistorted, while the remaining sixteen are foreshortened into rhombs. These sixteen squares are the faces of one $\{4, 4|4\}$.

When $l > 4$, the faces of $\{l, 4|n\}$ are simply all the $\{l\}$'s of the polytope* $t_{1,2}\{n, \frac{1}{2}l, n\}$. In the finite cases, this polytope can be described as the common content of two equal reciprocal regular polytopes $\{n, \frac{1}{2}l, n\}$. In particular, $t_{1,2}\{3, 3, 3\}$ or $t_{1,2}\alpha_4$ can also be described as a central section of the five-dimensional measure-polytope γ_5, analogous to the hexagonal section of the cube (γ_3). The general measure-polytope may be defined as the totality of points $(x_1, x_2, \ldots)$ for which $|x_i| \leqslant 1$; we take its section by the prime $\Sigma x_i = 0$. Thus the vertices of $t_{1,2}\alpha_4$, and so also of $\{6, 4|3\}$, are the thirty points

$$(1, 1, 0, -1, -1)$$

(permuted). (See Fig. x.) The polytope is bounded by ten truncated tetrahedra; the polyhedron separates these into two sets of five.

The regular 24-cell, $\{3, 4, 3\}$, may be defined as the totality of points for which

$$|x_i \pm x_j| \leqslant \sqrt{2}\dagger \quad (i, j = 1, 2, 3, 4;\ i < j).$$

Its reciprocal, with vertices $\pm(1, 1, 0, 0)$ (permuted), is then defined by

$$|x_i| \leqslant 1, \quad |x_1 \pm x_2 \pm x_3 \pm x_4| \leqslant 2.$$

Thus the vertices of $t_{1,2}\{3, 4, 3\}$, and so also of $\{8, 4|3\}$, are the 288 points

$$\pm(1, \sqrt{2}-1, \sqrt{2}-1, 3-2\sqrt{2}), \quad \pm(2\sqrt{2}-2, 2-\sqrt{2}, 2-\sqrt{2}, 0).$$

The polytope is bounded by forty-eight truncated cubes‡; the polyhedron separates these into two sets of twenty-four.

* Coxeter, **4**, 331.

† Or any other positive constant.

‡ Threlfall and Seifert, **1**, 63.

When $l = 4$ and $m > 4$, the polytope is $t_{0,3}\{n, \frac{1}{2}m, n\}$, which can be derived from the regular polytope $\{n, \frac{1}{2}m, n\}$ by uniformly shrinking all the bounding $\{n, \frac{1}{2}m\}$'s, and inserting n-gonal prisms between adjacent pairs of them*. The shrinkage is adjusted so that the lateral faces of the

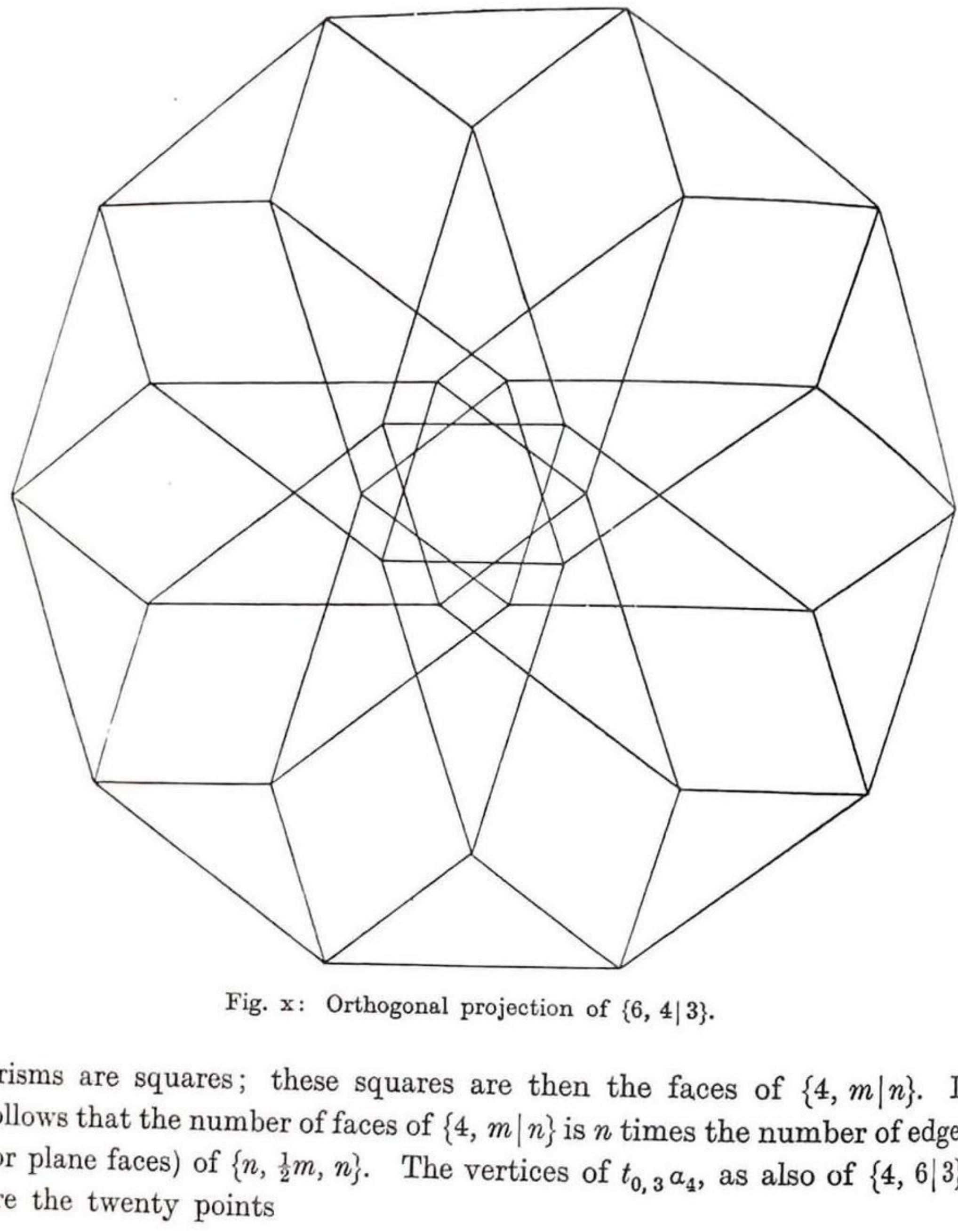


Fig. x: Orthogonal projection of $\{6, 4|3\}$.

prisms are squares; these squares are then the faces of $\{4, m|n\}$. It follows that the number of faces of $\{4, m|n\}$ is n times the number of edges (or plane faces) of $\{n, \frac{1}{2}m, n\}$. The vertices of $t_{0,3}\alpha_4$, as also of $\{4, 6|3\}$, are the twenty points

$$(1, 0, 0, 0, -1).$$

* Apart from a change of scale, this is equivalent to Mrs. Stott's "expansion" e_3. (Stott, 1, 9.)

The vertices of $t_{0,3}\{3, 4, 3\}$, as also of $\{4, 8|3\}$, are the 144 points

$$\pm(\sqrt{2}, 2-\sqrt{2}, 0, 0), \quad \pm(1, 1, \sqrt{2}-1, \sqrt{2}-1).$$

Any two adjacent faces of $\{l, m|n\}$ belong to a bounding $t_{0,1}\{\frac{1}{2}l, n\}$* of the corresponding polytope. The dihedral angle between them is therefore equal to the dihedral angle of the ordinary polyhedron† $\{\frac{1}{2}l, n\}$, namely

$$2 \arcsin \frac{\cos(\pi/n)}{\sin(2\pi/l)}.$$

In the three-dimensional cases, (2.3) shows that this is equal to

$$2 \arcsin \frac{\cos(\pi/m)}{\sin(\pi/l)} \quad \text{or} \quad 2 \arcsin \frac{\sin(\pi/m)}{\cos(\pi/l)}$$

according as the polyhedron $\{l, m|n\}$ is ordinary or skew. This explains the fact that the infinite skew polyhedra have the same dihedral angles as the ordinary polyhedra that are their "complements".

4. *The symmetry-groups.*

It is well known that the complete symmetry-group of the ordinary regular polyhedron $\{l, m\}$ is representable on a concentric sphere in such a way that the fundamental region is a spherical triangle of angles π/l, π/m, $\pi/2$. The group is generated by the reflections in the sides of this triangle, and has the abstract definition

$$R_1^2 = R_2^2 = R_3^2 = (R_1 R_2)^l = (R_2 R_3)^m = (R_1 R_3)^2 = 1.$$

The rotations

$$R = R_1 R_2, \quad S = R_2 R_3$$

generate a sub-group $\mathfrak{G}$, of order g, whose abstract definition is

$$R^l = S^m = (RS)^2 = 1. \tag{4.1}$$

This consists of all *rotations* that are symmetries of the polyhedron. The complete group is derivable from it by adjoining a single reflection, and so is of order $2g$. The area of the fundamental region is $(1/l+1/m-\frac{1}{2})\pi$. Since $2g$ such regions exactly cover the sphere, we have

$$2/g = 1/l+1/m-\tfrac{1}{2}. \tag{4.2}$$

---

* For $l = 4$, an n-gonal prism.

† For $l = 4$, of the n-gon itself.

Instead of representing the group on a sphere, we can just as well represent it on the surface of the polyhedron itself. The fundamental region is now a plane right-angled triangle $O_1 O_2 O_3$, where

O_3 is the centre of a face of the polyhedron,

O_2 is the mid-point of a side of this face, and

O_1 is one end of this side.

R_1, R_2, R_3 are reflections in the planes joining the centre of the polyhedron to the sides $O_2 O_3$, $O_3 O_1$, $O_1 O_2$ of this triangle. Every face can be divided into $2l$ such triangles, and then each edge of the polyhedron belongs to four of them, and each vertex to $2m$. It is convenient to imagine the triangles to be coloured alternately white and black, so that there are l of either colour in each face, two of either colour at each edge, m of either colour at each vertex, and g of either colour altogether. Hence, if the polyhedron has f faces, e edges, and v vertices,

$$lf = 2e = mv = g,$$

i.e.

(4.3) $$f = g/l, \quad e = g/2, \quad v = g/m.$$

Thus (4.2) is equivalent to Euler's theorem

(4.4) $$f + v - e = 2.$$

The faces of a regular *skew* polyhedron can be divided into right-angled triangles in the same manner. Any such triangle is still a fundamental region for the complete symmetry group, but the generating operations which transform the first triangle into its neighbours need no longer be reflections. Actually, R_1 and R_3 still are reflections (in the planes or three-spaces that we called λ and μ), but R_2 is the rotation through π about the hypotenuse $O_1 O_3$ (or about the plane joining this line to the centre). Hence the sub-group generated by R and S no longer consists solely of rotations. However, since it transforms white triangles into white and black into black, we shall call it the *intrinsic rotation-group* of $\{l, m \mid n\}$, and still denote it by $\mathfrak{G}$, and its order by g. Both groups now require an extra relation in their abstract definitions, on account of the multiple connectivity of the surface. The formulae (4.3) still hold, but (4.4) has to be replaced by

$$f + v - e = 2 - 2p,$$

where p is the genus; consequently (4.2) becomes

(4.5) $$p = \tfrac{1}{2}g(\tfrac{1}{2}-1/l-1/m)+1.$$

This enables us to find p when g is known, but we have no simple expression for g in terms of l, m, n.

Actually, we can find g in any given case by considering the group generated by the reflections in the planes or three-spaces λ, λ', μ, μ'. When we denote these reflections by L, L', M, M', (2.1) shows that this group is defined by

(4.6)* $$L^2 = L'^2 = M^2 = M'^2 = (LM)^2 = (L'M')^2$$
$$= (LL')^{\frac{1}{2}l} = (MM')^{\frac{1}{2}m} = (LM')^n = (L'M)^n = 1.$$

Its fundamental region includes two of our right-angled triangles, namely one white and one black, with a common hypotenuse. Hence it is a subgroup of index 2 in the complete group of the polyhedron, so that its order is g†. The rotation R_2, which interchanges the white and black triangles, is a symmetry of the tetrahedron $\lambda\lambda'\mu\mu'$, and transforms L, L', M, M' into L', L, M', M, respectively. Thus we have

$$L = R_1, \quad L' = R_2 R_1 R_2, \quad M = R_3, \quad M' = R_2 R_3 R_2.$$

The complete group is derivable from (4.6) by adjoining the involutory operation R_2. By direct substitution, its abstract definition is thus

(4.7) $$R_1{}^2 = R_2{}^2 = R_3{}^2 = (R_1 R_2)^l = (R_2 R_3)^m = (R_1 R_3)^2$$
$$= (R_1 R_2 R_3 R_2)^n = 1.$$

Finally, the intrinsic rotation-group $\mathfrak{G}$ can be derived from this (by writing $R_1 R_2 = R$, $R_2 R_3 = S$) in the form

(4.8) $$R^l = S^m = (RS)^2 = (RS^{-1})^n = 1.$$

The fact that these relations suffice to define $\mathfrak{G}$ may be verified by adjoining the involutory operation R_2 (which transforms R and S into their inverses), and so reconstructing (4.7).

In the case of $\{4, 4 \mid n\}$, the group generated by reflections is $[n, 2, n]$ or $[n]\times[n]$, the direct product of two dihedral groups of order $2n$. Hence

---

* Coxeter, 3.

† Therefore g is infinite when $2\sin(\pi/l)\sin(\pi/m) < \cos(\pi/n)$ and l, m are even.

$\mathfrak{G}$, defined by*

$$R^4 = S^4 = (RS)^2 = (RS^{-1})^n = 1,$$

is likewise of order $4n^2$. It has a simple representation as a permutation-group of degree $2n$, namely

$$R = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8)(\ldots\ 2n), \quad S = (1\ 2)(3\ 4\ 5\ 6)(\ldots\ 2n);$$

e.g., when $n = 3$†,

$$R = (1\ 2\ 3\ 4)(5\ 6), \qquad S = (1\ 2)(3\ 4\ 5\ 6),$$

and when $n = 4$‡,

$$R = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8), \qquad S = (1\ 2)(3\ 4\ 5\ 6)(7\ 8).$$

In the case of $\{4, 6 \mid 3\}$ or $\{6, 4 \mid 3\}$, the group generated by reflections is $[3, 3, 3]$, the symmetric group of order 120. Hence $\mathfrak{G}$, defined by

$$R^4 = S^6 = (RS)^2 = (RS^{-1})^3 = 1,$$

is likewise of order 120; and by considering the permutations (1 4 2 5), (1 5) (2 3 4), we see that it is again the symmetric group. Geometrically, $[3, 3, 3]$ permutes the five vertices of the regular simplex α_4. $\mathfrak{G}$ differs from it in having each *odd* permutation combined with the central inversion.

In the case of $\{4, 8 \mid 3\}$ or $\{8, 4 \mid 3\}$, the group generated by reflections is $[3, 4, 3]$, of order 1152§. Hence $\mathfrak{G}$, defined by

$$R^4 = S^8 = (RS)^2 = (RS^{-1})^3 = 1,$$

is likewise of order 1152 (although it is a quite different group). The operation

$$(R^2 S^2)^6 = (R_1 . R_2 R_1 R_2 . R_2 R_3 R_2 . R_3)^6 = (LL' M' M)^6$$

is the central inversion; and, by considering the permutations

$$(1\ 8)\ (2\ 7\ 3\ 6)\ (4\ 5), \quad (1\ 6\ 3\ 7\ 4\ 5\ 2\ 8),$$

* Burnside, 1, 419 (III; $a = n$, $b = 0$).

† Miller, 1, 368 (Degree Six, Order 36, No. 2). The relation $(s_1^2 s_2^2)^3 = 1$ is superfluous.

‡ Burns, 1, 208 (Order 64, No. 4). The relation $(s_1 s_2^2)^4 = 1$ is superfluous.

§ Goursat, 1, 87.

we can identify the central quotient group

$$R^4 = S^8 = (RS)^2 = (RS^{-1})^3 = (R^2 S^2)^6 = 1$$

with Dr. Burns's group of order 576, No. 3*.

5. *Topological extension of the theory: the general polyhedron* $\{l, m \mid n\}$.

Every regular polyhedron can be interpreted topologically as a *regular map* in the sense of Brahana†, the faces of the polyhedron being the countries of the map. If the faces are l-gons, m at each vertex, the map may be built up, face by face, by lettering the vertices, and calling the faces ABC..., ABE..., and so on. *E.g.*, when $l = 4$ and $m = 3$, we have the "topological cube"

ABCD, ABEF, ADGF, BCHE, CDGH, EFGH.

If this abstract construction is carried on in such a way that new letters are introduced whenever possible (*i.e.*, using only the number of sides of the face and the number of faces at a vertex), then the map is automatically regular, and forms a simply-connected surface, finite or infinite according as

$$1/l + 1/m > \text{ or } \leqslant \tfrac{1}{2}.$$

It can be metrically realized as a partition of the sphere, or of the Euclidean or hyperbolic plane, into regular l-gons; hence no confusion can arise by using the symbol $\{l, m\}$.

Any other regular map of l-gons, m at each vertex, can be derived from $\{l, m\}$ by *identifying* certain edges. The intrinsic rotation-group $\mathfrak{G}$, of order $g = lf = 2e = mv$, may be defined topologically as a permutation group on the edges of the map; namely, as generated by R, which cyclically permutes the edges of one face, and S, which cyclically permutes the edges at one vertex‡. It is clearly a factor group of the rotation-group of $\{l, m\}$, whose abstract definition is (4.1). The identification of a pair of edges of $\{l, m\}$ can at once be interpreted as an extra relation between R and S, and will necessitate the identification of every other pair that are similarly related.

We may, for instance, identify two edges belonging to a chain of edges which leaves, at each vertex, two faces on the left (and $m-2$ on the right).

---

* Burns, 1, 210. The central quotient group of [3, 4, 3], on the other hand, is No. 2.

† Brahana, 1, 269.

‡ Our generators R, S differ trivially from Brahana's S, T; in fact, our R, S are his S, $S^{-1}T$, while his S, T are our R, RS. Also our l, f are his k, n.

It is easily seen (Fig. xi*) that the operation RS^{-1} takes us one step along such a chain. Thus the relation

$$(RS^{-1})^n = 1$$

corresponds to the identification of edges which differ by n steps.

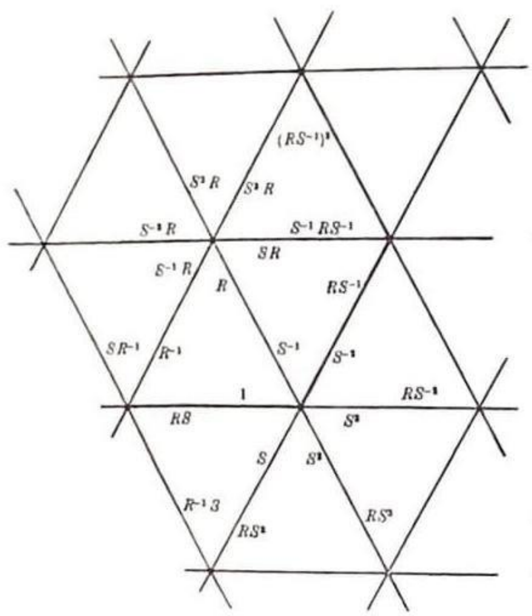


Fig. xi: $R^3 = S^6 = (RS)^2 = 1$.

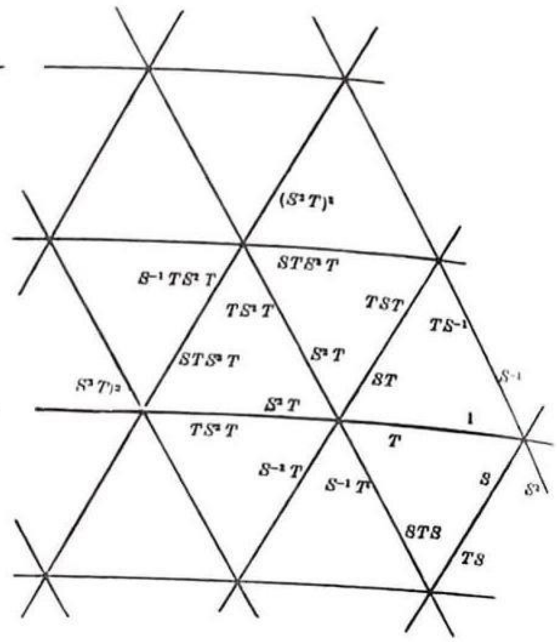


Fig. xii: $S^6 = T^2 = (ST)^3 = 1$.

We now have an interpretation for the symbol $\{l, m \mid n\}$ regardless of the parity of l and m. §4 shows that this agrees with our previous interpretation in the case when l and m are both even.

In a family of abstract groups with various periods assigned to certain fixed combinations of generators, it usually happens that the greater periods correspond to greater order†. Accordingly, we should expect the criterion

$$(5.1) \qquad 2 \sin(\pi/l) \sin(\pi/m) > \cos(\pi/n)$$

(which precisely determines the finiteness of the polyhedron $\{l, m \mid n\}$

---

* In the special case when $l = 3$, (4.1) implies $(RS^{-1})^m = 1$. But when l and m are both greater than 3, the period of RS^{-1} is infinite. This follows from (1.1), by considering the Minkowskian polyhedron $\{l, m\}$; for, if $l \geqslant 4$ and $m \geqslant 4$, then $\cos(\pi/n) \geqslant 1$, and π/n is a hyperbolic angle.

† This is merely a working rule, and not a demonstrable theorem, as the following example shows: although $\{6, 6 \mid 3\}$ is infinite, we shall see later that $\{6, 7 \mid 3\}$ and $\{7, 7 \mid 3\}$ are finite.

when l and m are even) to retain some significance when the parity of l and m is unrestricted. Actually, we find that it fails in two respects.

First, for certain values of l, m, n, satisfying the criterion (as well as for certain values violating it), the group *collapses*, in the sense that the relations (4.8) imply $R = S = 1$. *E.g.*, there is no polyhedron $\{4, 5 \mid 3\}$. Secondly, for certain values of l, m, n, violating the criterion, the group remains finite. In fact, the order of the group, considered as a function of l, m, n, has a tendency to take, when l or m is odd, a smaller value than that which we should expect by interpolation from the values when l and m are even. (Since the value when l and m are even is determined by the volume of a spherical tetrahedron, this interpolation could be given a precise meaning in terms of Schläfli functions.) However, we can say this: *whenever the group is infinite, the criterion is violated, i.e.*, whenever the criterion is satisfied, the group either is finite or collapses. In Figs. vi and vii, we mark the known finite polyhedra as O's, the known infinite polyhedra as dots, and the known cases of collapse as crosses. We observe that the O's are all inside or "just outside" the region of validity of the criteria. (Unmarked lattice points correspond to groups which have not yet been investigated.)

When l or $m = 3$, we must have $n = m$ or l (respectively); any other value of n causes collapse (although, if the assigned value of n is a multiple of its proper value, the collapse will be merely partial).

When $m > 3$, and l and n are both even (or when $l > 3$, and m and n are both even), *the criterion* (5.1) *holds perfectly*; *e.g.*, $\{4, 7 \mid 4\}$ and $\{5, 6 \mid 4\}$ are infinite. For (4.8) can be written in the form

$$S^m = T^2 = (ST)^l = (S^2 T)^n = 1, \tag{5.2}$$

which has been thoroughly investigated elsewhere* in the case when l and n are even.

By putting S^2 in place of S in (5.2), we see that *$\{l, 5 \mid n\}$ and $\{n, 5 \mid l\}$ have the same group*. In particular, $\{5, 5 \mid 3\}$ has the same group as $\{3, 5 \mid 5\}$ or $\{3, 5\}$, namely the icosahedral group. $\{5, 5 \mid 3\}$ can, in fact, be realized metrically as the *great dodecahedron*† $\{5, \frac{5}{2}\}$ (Fig. xiii). All other polyhedra $\{n, 5 \mid 3\}$, $\{5, n \mid 3\}$ are impossible, since they would have the same group as $\{3, 5 \mid n\}$.

* Coxeter, 5, 284.

† *Cf.* Brahana and Coble, 1, 14 (Fig. VII), The other three Kepler-Poinsot polyhedra give nothing fresh. In fact, $\{\frac{5}{2}, 5\}$, $\{\frac{5}{2}, 3\}$, $\{3, \frac{5}{2}\}$, are topologically identical with $\{5, \frac{5}{2}\}$, $\{5, 3\}$, $\{3, 5\}$, respectively.

Apart from

$$\{3,\, m\,|\,n\},\quad \{m,\, 3\,|\,n\}\quad (n \neq m),$$

$$\{n,\, 5\,|\,3\},\quad \{5,\, n\,|\,3\}\quad (n \neq 5),$$

there are no known cases of collapse.

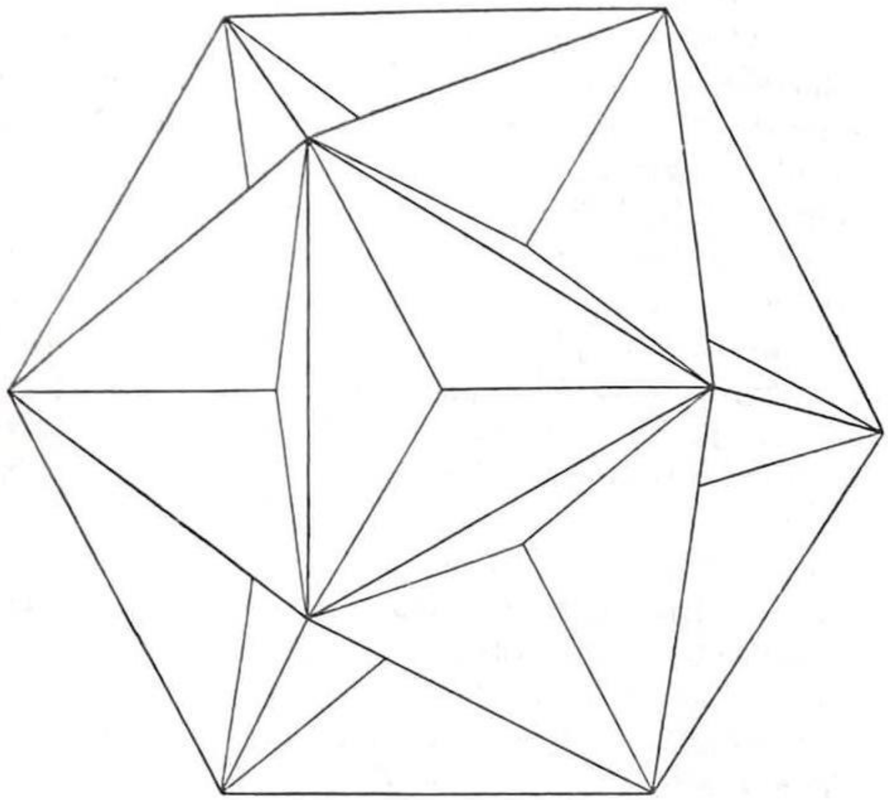


Fig. xiii: The great dodecahedron, $\{5,\, 5\,|\,3\}$.

The remaining polyhedra that satisfy (5.1) are

$$\{4,\, 5\,|\,4\},\quad \{4,\, 7\,|\,3\},\quad \{4,\, 5\,|\,5\},$$

and their reciprocals. The last of these has, of course, the same group as $\{5,\, 5\,|\,4\}$, which does not satisfy (5.1).

For $\{4,\, 5\,|\,4\}$ and $\{5,\, 4\,|\,4\}$, $\mathfrak{G}$ is*

$$S^5 = T^2 = (ST)^4 = (S^2\,T)^4 = 1,$$

of order 160. $\{4,\, 5\,|\,4\}$ can be realized metrically in five dimensions, by taking half the squares of the measure-polytope γ_5. Fig. xiv shows an isometric projection of the vertices and edges of γ_5, in which the 80 squares appear as rhombs of two kinds: 40 of angle $\frac{1}{5}\pi$, and 40 of angle $\frac{2}{5}\pi$. Either

* Coxeter, **5**, 284.

set of 40 are projections of the faces of a $\{4, 5|4\}$. (This partition of the 80 squares is not really unique, but depends on the plane of projection.)

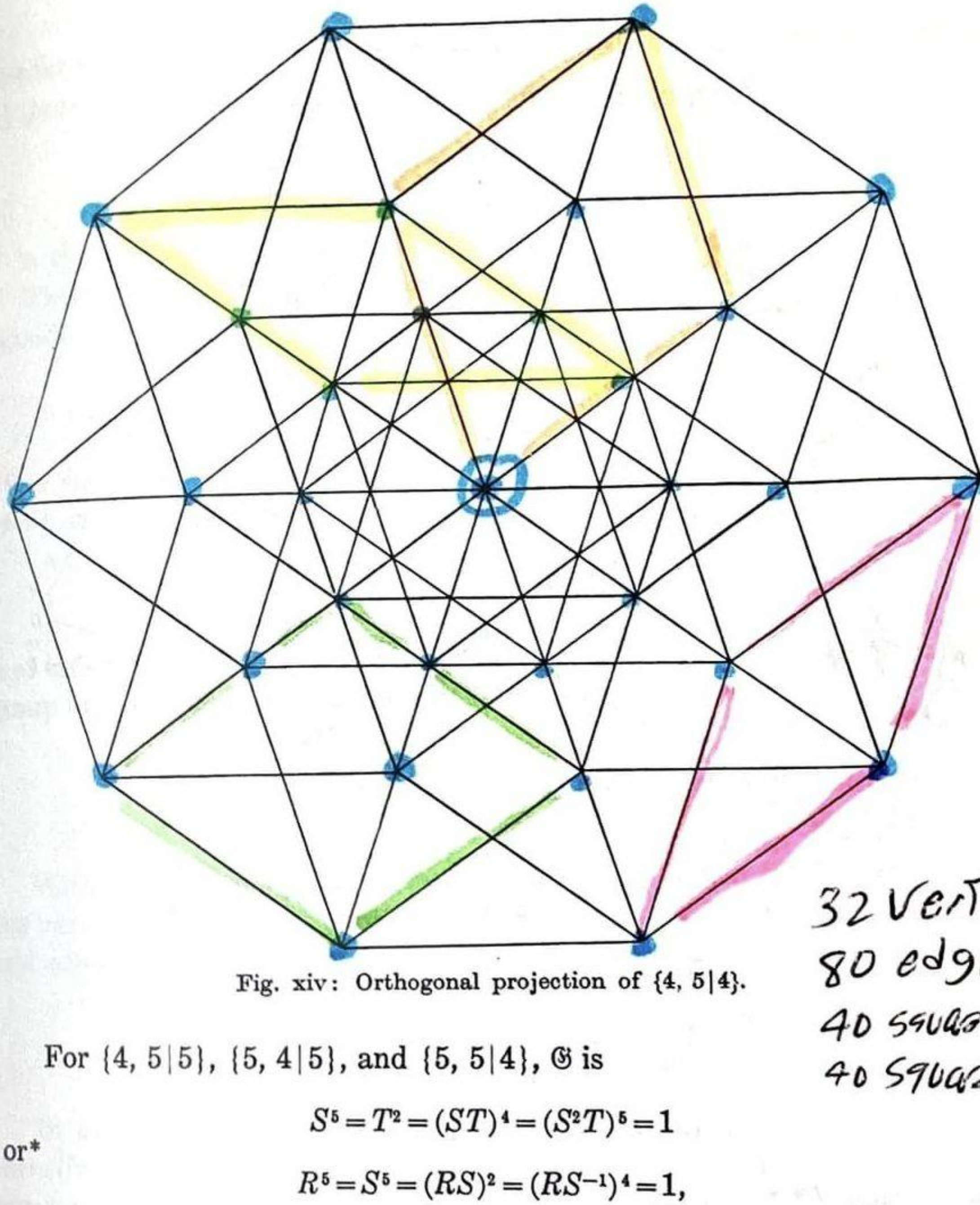


Fig. xiv: Orthogonal projection of $\{4, 5|4\}$.

32 vertices
80 edges
40 square faces
40 square holes

For $\{4, 5|5\}$, $\{5, 4|5\}$, and $\{5, 5|4\}$, $\mathfrak{G}$ is

$$S^5 = T^2 = (ST)^4 = (S^2T)^5 = 1$$

or*

$$R^5 = S^5 = (RS)^2 = (RS^{-1})^4 = 1,$$

the simple group of order 360 (*i.e.*, the alternating group of degree six). Fig. xv shows a conformal representation of $\{4, 6\}$ *qua* partition of the hyperbolic plane, with letters to indicate the identifications that produce $\{4, 6|3\}$ (*cf.* Fig. viii).

---

*Todd and Coxeter, **1**, 31 (5); Brahana, **1**, 274.

For $\{4, 7|3\}$ and $\{7, 4|3\}$, $\mathfrak{G}$ is*

$$S^7 = T^2 = (ST)^4 = (S^2T)^3 = 1,$$

the simple group of order 168. This is a special case of

$$S^7 = T^2 = (ST)^l = (S^2\,T)^3 = 1.$$

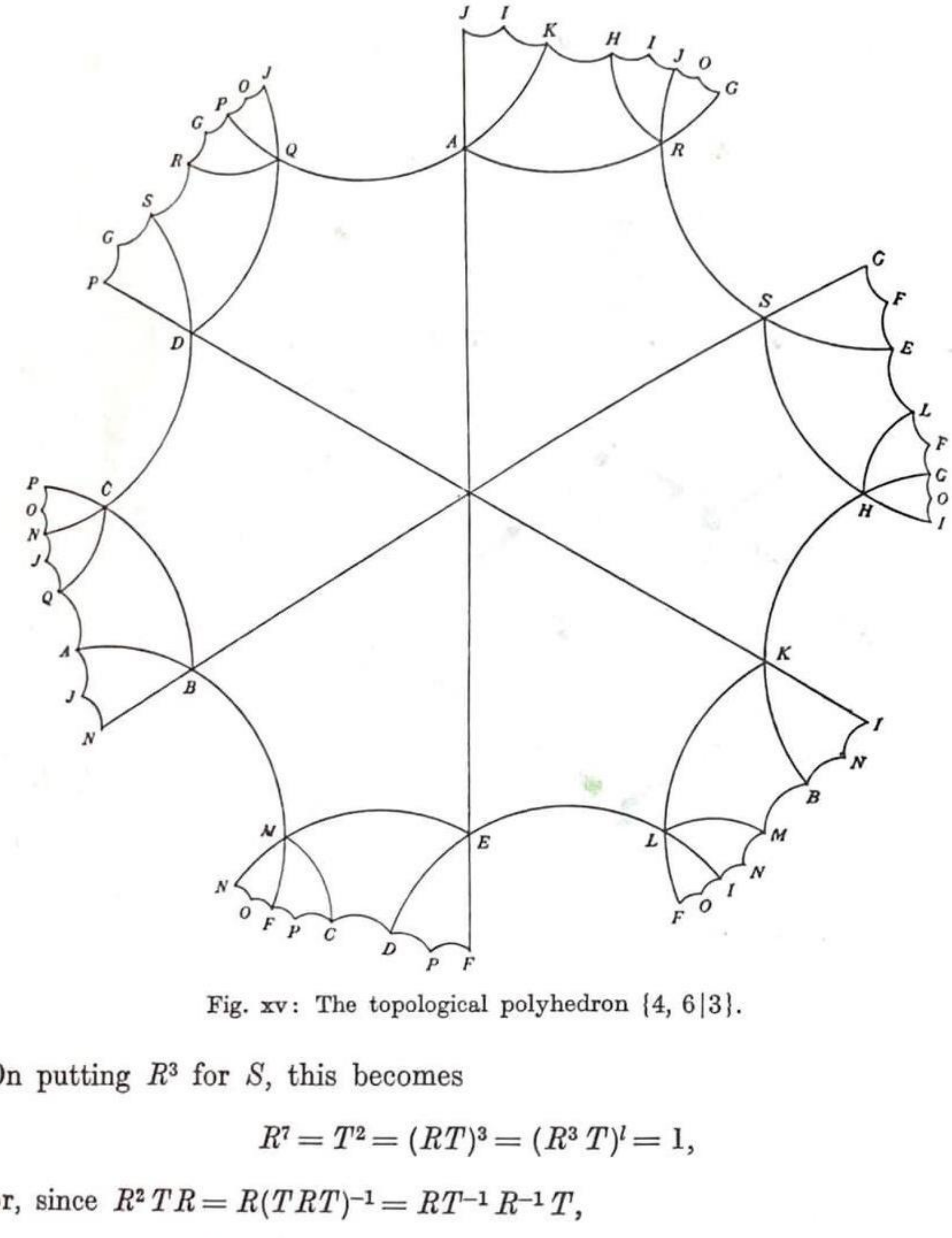


Fig. xv: The topological polyhedron $\{4, 6|3\}$.

On putting R^3 for S, this becomes

$$R^7 = T^2 = (RT)^3 = (R^3\,T)^l = 1,$$

or, since $R^2\,TR = R(TRT)^{-1} = RT^{-1}\,R^{-1}\,T$,

$$R^7 = T^2 = (RT)^3 = (RT^{-1}\,R^{-1}\,T)^l = 1.$$

*Burnside, **1**, 422. $S_7 = S^2$, $S_2 = T$.

In the form

$$S^3 = T^2 = (ST)^7 = (S^{-1}T^{-1}ST)^l = 1,$$

this family of groups has been extensively studied by Brahana and Sinkov*. By putting $l = 6$ and then $l = 7$, we see that, for the three polyhedra

$$\{6, 7 \mid 3\}, \quad \{7, 6 \mid 3\}, \quad \{7, 7 \mid 3\},$$

$\mathfrak{G}$ is the simple group of order 1092.

There remain, in Fig. vi, one pair of lattice points that are "only just outside the region", namely the points corresponding to the polyhedra

$$\{4, 9 \mid 3\}, \quad \{9, 4 \mid 3\}.$$

After enormous labour, I succeeded in enumerating the 272 co-sets of a cyclic subgroup of order 9, thus proving that the group

$$S^9 = T^2 = (ST)^4 = (S^2T)^3 = 1$$

is of order 2448. Then Sinkov completed its identification with the simple group of that order, by citing the linear fractional substitutions

$$S = \left(7\,\frac{x+1}{x+4}\right), \qquad T = (-x) \quad (\text{mod } 17).$$

With the help of (4.3), we can now write out Table I. In describing the various groups, we use the symbols S_n and A_n to denote the symmetric and alternating groups of degree n.

6. *A further extension: the polyhedron* $\{q_1, m \mid q_2, q_3, \ldots\}$.

In §5, we derived the topological polyhedron $\{l, m \mid n\}$ from $\{l, m\}$ by identifying two edges that belong to a certain "chain". We may generalize our results by using a chain of edges which leaves, at each vertex, (say) j on the right (and $m-j$ on the left). When $j = 1$ this is a face; when $j = 2$ it is a "hole". We define a generalized skew polyhedron

$$\{q_1, m \mid q_2, q_3, \ldots\} \tag{6.1}$$

---

* Brahana, 2; Sinkov, 1.

by identifying points q_j steps along the j-th chain. In the most general case we would allow j to take all values from 1 to $[\frac{1}{2}m]$. In certain cases, such a skew polyhedron can be realized metrically in Euclidean space of m (or fewer) dimensions. Its vertex figure is then a skew m-gon whose sides, first diagonals, second diagonals, etc., are of lengths

$$2\cos(\pi/q_1), \quad 2\cos(\pi/q_2), \quad 2\cos(\pi/q_3), \quad \ldots.$$

Since these generalized skew polyhedra do not necessarily occur in reciprocal pairs, it is convenient to replace R by TS^{-1}, as in (5.2). A glance at Fig. xii makes it clear that the group takes the form

$$S^m = T^2 = (S^j T)^{q_j} = 1 \quad (j = 1, 2, \ldots),$$

or, in terms of R and S,

$$R^{q_1} = S^m = (RS)^2 = (RS^{-1})^{q_2} = (RS^{-2})^{q_3} = \ldots = 1.$$

When all the q's are even, the only finite polyhedra of this kind are*

$$\{4, m \mid 4^{[\frac{1}{2}m]-1}\}$$

and

$$\{4, m \mid 4^{\frac{1}{2}m-2}, 2p\} \quad (m \text{ even}).$$

These can both be realized metrically in Euclidean m-space, the former by squares from the measure-polytope γ_m, and the latter by squares from the generalized prism (or "rectangular product") of $\frac{1}{2}m$ $2p$-gons:

$$[\{2p\}, \{2p\}, \ldots] \quad \text{or} \quad \{2p\}^{\frac{1}{2}m}.$$

The polyhedron $\{4, m \mid 4^{[\frac{1}{2}m]-1}\}$, with $g = 2^m m$, has $2^{m-2}m$ faces (squares), $2^{m-1}m$ edges, and 2^m vertices (viz., all the edges and vertices of γ_m). Accordingly, its genus is $2^{m-3}(m-4)+1$. (The cases $m=4$ and $m=5$ were considered earlier. See Figs. ix, xiv.) The properties of $\{4, m \mid 4^{\frac{1}{2}m-2}, 2p\}$ (m even) can be deduced similarly from the fact that $g = (2p)^{\frac{1}{2}m} m$.

The vertex figure of $\{4, m \mid 4^{[\frac{1}{2}m]-1}\}$ is a skew m-gon whose sides and all kinds of diagonals are of length $\sqrt{2}$. In fact, the vertex figure of γ_m is a regular simplex $\alpha_{m-1}\sqrt{2}$, whose Petrie polygon† is the vertex figure of the

* Coxeter, 5, 283, (4.2), (4.5). We write 4^n for a row of n 4's.

† Coxeter, 1, 203.

polyhedron. The vertex figure of $\{4, m\,|\,4^{\frac{1}{2}m-2}, 2p\}$ (m even) differs from this in having all pairs of *opposite* vertices distant $2 \cos(\pi/2p)$. When we make p infinite, this becomes the Petrie polygon of the cross-polytope $\beta_{\frac{1}{2}m}\sqrt{2}$. Now, this cross-polytope is the vertex figure of $\delta_{\frac{1}{2}m+1}$, the net of measure-polytopes, *i.e.*, the infinite polytope whose vertices are all the lattice points in $\frac{1}{2}m$ dimensions. Hence, writing r for $\frac{1}{2}m$, we have the infinite skew polyhedron

$$\{4,\ 2r\,|\,4^{r-2}\},$$

whose faces belong to δ_{r+1}. This can be regarded as a generalization of $\{4, 4\}$ and $\{4, 6\,|\,4\}$.

There is also a generalization of $\{6, 3\}$ and $\{6, 4\,|\,4\}$: the infinite polyhedron

$$\{6,\ m\,|\,4^{[\frac{1}{2}m]-1}\}.$$

Its faces are all the hexagons of the uniform $(m-1)$-dimensional space-filling* whose graph consists of an m-gon with every vertex ringed. The remaining plane faces of this space-filling are squares, which form, when $m = 5$, another infinite polyhedron:

$$\{4,\ 5\,|\,6\}.$$

When the parity of the q's is unrestricted, very little is known about the possible polyhedra (6.1). One line of investigation is suggested by the fact that

$$\{6,\ 6\,|\,3,\ 4\}$$

can be realized metrically by taking all the equatorial hexagons of all the bounding cuboctahedra of the "truncated tessaract" $t_1\gamma_4$. The corresponding group is simply

$$S^6 = T^2 = (S^2\,T)^3 = (S^3\,T)^4 = 1,$$

since these relations imply $(ST)^6 = 1$. This is the "hyper-pyritohedral" group† $[(3, 3)', 4]$, of order 192, since it is derivable from $[3, 3, 4]$‡ by putting

$$S = R_1\,R_2\,R_4, \quad T = R_1\,R_3.$$

---

* Coxeter, **4**, 334.

† Coxeter, **5**, 295.

‡ *I.e.*,

$$R_1^2 = R_2^2 = R_3^2 = R_4^2 = (R_1\,R_2)^3 = (R_2\,R_3)^3 = (R_3\,R_4)^4 = (R_1\,R_3)^2 = (R_1\,R_4)^2 = (R_2\,R_4)^2 = 1.$$

The hexagons cut one another diagonally, producing "false vertices". This polyhedron, therefore, is a kind of generalized Kepler-Poinsot polyhedron. Four-dimensional space admits nineteen reciprocal pairs of such skew star-polyhedra; but they lie beyond the scope of the present work.

In conclusion, we mention the case when none of the q's are specified save q_1 and q_3, *i.e.*, the case of the polyhedron*

$$\{l, m|, q\} \quad (m \geqslant 6)$$

with q-gonal "second holes". The group is

$$S^m = T^2 = (ST)^l = (S^3 T)^q = 1, \tag{6.2}$$

or

$$R^l = S^m = (RS)^2 = (RS^{-2})^q = 1,$$

or

$$R^l = T^2 = (RT)^m = (R^2 TRT)^q = 1.$$

When $l = 3$, this last definition becomes†

$$R^3 = T^2 = (RT)^m = (R^{-1} TRT)^q = 1.$$

In particular, one of Brahana's results‡ shows that $\{3, 8|, 6\}$ is infinite.

When $m = 7$, we write S^2 for S in (6.2), obtaining

$$S^7 = T^2 = (ST)^q = (S^2 T)^l = 1.$$

Thus $\{l, 7|, q\}$ *has the same group as* $\{q, 7|l\}$. $\{3, 7|, 6\}$ and $\{3, 7|, 7\}$ provide the extraordinary phenomenon of two polyhedra which have the same number of vertices, edges, and faces (and therefore the same genus), *and the same group*, but which still are distinct.

When m is even, R and S^2 generate a self-conjugate sub-group of index 2:

$$R^l = S'^{\frac{1}{2}m} = (RS')^l = (RS'^{-1})^q = 1,$$

so that, when we put $s_1 = R^{-1}$, $s_2 = RS'$,

$$s_1^{\,l} = s_2^{\,l} = (s_1 s_2)^{\frac{1}{2}m} = (s_1^{\,2} s_2)^q = 1.$$

---

* $m = 3$ implies $q = 2$; $m = 4$ implies $q = l$; and $\{l, 5|, q\}$ is the same as $\{l, 5|q\}$.

† Brahana, 2.

‡ Brahana, 3, 901.

It seems likely that the only non-trivial cases when this group is finite are those considered by G. A. Miller*, and one immediate consequence of these, namely:

$$l = 3, \quad \tfrac{1}{2}m = q = 4,$$

$$l = 3, \quad \tfrac{1}{2}m = 4, \quad q = 5,$$

$$l = 3, \quad \tfrac{1}{2}m = 5, \quad q = 4,$$

$$l = 4, \quad \tfrac{1}{2}m = q = 3.$$

In the first case, Miller gives the permutations

$$s_1 = (abc)\,(def), \quad s_2 = (aeh)\,(cdg).$$

Since S transforms s_1 into s_2, and $S^2 = S' = s_1 s_2$, we have

$$R = (acb)\,(dfe), \quad S = (agbcdehf).$$

Hence the group of $\{3, 8\,|\,, 4\}$ is the "extended 168-group"†, *i.e.*, the group of all linear fractional transformations modulo 7. By putting $s_1' = S'$, $s_2' = S'^{-1} R$, we obtain the 168-group itself in the alternative form

$$s_1'^4 = s_2'^4 = (s_1' s_2')^3 = (s_1'^2 s_2')^3 = 1,$$

with $$s_1' = (abdh)\,(cefg), \quad s_2' = (ahfd)\,(bcge).$$

In this case, s_1' is transformed into s_2' by $(afcebd)$; so the group of $\{4, 6\,|\,, 3\}$ is again the extended 168-group.

Du Val has pointed out that the polyhedron $\{3, 7\,|\,, 4\}$ can be metrically realized (albeit singularly) in seven dimensions. Its faces are the fifty-six triangles of the regular simplex α_7; its edges and vertices are those of the simplex, each counted three times. The six triangles that meet at an edge of α_7 have to be paired in a special manner.

---

* Miller, 2.

† Van der Waerden calls this group *PGL* (2, 7), *PSL* (2, 7) being the 168-group itself, which Dickson calls *LF*(2, 7).

TABLE I. Finite polyhedra $\{l, m|n\}$.

Polyhedron	f	e	v	p	$\mathfrak{G}$	g
$\{3, 3\|3\} = \{3, 3\}$	4	6	4	0	$A_4 \sim LF(2, 3)$	12
$\{3, 4\|4\} = \{3, 4\}$	8	12	6	0	S_4	24
$\{4, 3\|4\} = \{4, 3\}$	6	12	8	0		
$\{3, 5\|5\} = \{3, 5\}$	20	30	12	0	$A_5 \sim LF(2, 5)$	60
$\{5, 3\|5\} = \{5, 3\}$	12	30	20	0		
$\{5, 5\|3\} = \{5, \frac{5}{2}\}$	12	30	12	4		
$\{4, 4\|n\}$	n^2	$2n^2$	n^2	1		$4n^2$
$\{4, 5\|4\}$	40	80	32	5		160
$\{5, 4\|4\}$	32	80	40	5		
$\{4, 6\|3\}$	30	60	20	6	S_5 A_4	120
$\{6, 4\|3\}$	20	60	30	6		
$\{4, 7\|3\}$	42	84	24	10	$LF(2, 7)$	168
$\{7, 4\|3\}$	24	84	42	10		
$\{4, 5\|5\}$	90	180	72	10	A_6	360
$\{5, 4\|5\}$	72	180	90	10		
$\{5, 5\|4\}$	72	180	72	19		
$\{4, 8\|3\}$	288	576	144	73	F_4	1152
$\{8, 4\|3\}$	144	576	288	73		
$\{6, 7\|3\}$	182	546	156	105	$LF(2, 13)$	1092
$\{7, 6\|3\}$	156	546	182	105		
$\{7, 7\|3\}$	156	546	156	118		
$\{4, 9\|3\}$	612	1224	272	171	$LF(2, 17)$	2448
$\{9, 4\|3\}$	272	1224	612	171		
$\{7, 8\|3\}$	1536	5376	1344	1249		10752
$\{8, 7\|3\}$	1344	5376	1536	1249		

TABLE II. Finite polyhedra $\{l, m|, q\}$.

Polyhedron	f	e	v	p	$\mathfrak{G}$	g
$\{3, 6\|, q\}$	$2q^2$	$3q^2$	q^2	1		$6q^2$
$\{3, 2q\|, 3\}$	$2q^2$	$3q^2$	$3q$	$\frac{1}{2}(q-1)(q-2)$		$6q^2$
$\{3, 7\|, 4\}$	56	84	24	3	$LF(2, 7)$	168
$\{3, 8\|, 4\}$	112	168	42	8	$PGL(2, 7)$	336
$\{4, 6\|, 3\}$	84	168	56	15		
$\{3, 7\|, 6\}$	364	546	156	14	$LF(2, 13)$	1092
$\{3, 7\|, 7\}$	364	546	156	14		
$\{3, 8\|, 5\}$	720	1080	270	46		2160
$\{3, 10\|, 4\}$	720	1080	216	73		
$\{4, 6\|, 2\}$	12	24	8	3	$S_4 \times S_2$	48
$\{5, 6\|, 2\}$	24	60	20	9	$A_5 \times S_2$	120
$\{3, 11\|, 4\}$	2024	3036	552	231	$LF(2, 23)$	6072
$\{3, 7\|, 8\}$	3584	5376	1536	129		10752
$\{3, 9\|, 5\}$	12180	18270	4060	1016	$LF(2, 29) \times A_3$	36540

References.

Andreini, 1. " Sulle reti di poliedri regolari e semiregolari ", *Mem. della Soc. ital. delle Sci.* (3), 14 (1905), 75–110.

Brahana, 1. " Regular maps and their groups ", *American J. of Math.*, 49 (1927), 268–284.

———, 2. " Certain perfect groups generated by two operators of orders two and three ", *American J. of Math.*, 50 (1928), 345–356.

———, 3. " On the groups generated by two operators of orders two and three whose product is of order eight ", *American J. of Math.*, 53 (1931), 891–901.

Brahana and Coble, 1. " Maps of twelve countries with five sides with a group of order 120 containing an ikosahedral subgroup ", *American J. of Math.*, 48 (1926), 1–20.

Burns, 1. " Abstract definitions of groups of degree eight ", *American J. of Math.*, 37 (1915), 195–214.

Burnside, 1. " Theory of groups of finite order " (Cambridge, 1911).

Cauchy, 1. " Recherches sur les polyèdres ", *Journal de l'Ecole Polyt.*, 9 (1813), 68–86.

Coxeter, 1. " The densities of the regular polytopes ", *Proc. Camb. Phil. Soc.*, 27 (1931), 201–211.

———, 2. " The polytopes with regular-prismatic vertex figures ", *Proc. London Math. Soc.* (2), 34 (1932), 126–189.

———, 3. " The complete enumeration of finite groups of the form

$$R_i^2 = (R_i R_j)^{k_{ij}} = 1 ",$$

Journal London Math. Soc., 10 (1935), 21–25.

———, 4. " Wythoff's construction for uniform polytopes ", *Proc. London Math. Soc.* (2), 38 (1935), 327–339.*

———, 5. " The abstract groups $R^m = S^m = (R^j S^j)^{p_j} = 1$, $S^m = T^2 = (S^j T)^{2p_j} = 1$, and $S^m = T^2 = (S^{-j} T S^j T)^{p_j} = 1$ ", *Proc. London Math. Soc.* (2), 41 (1936), 278–301.

Goursat, 1. " Sur les substitutions orthogonales et les divisions régulières de l'espace ", *Ann. Sci. de l'Ecole Norm. Sup.* (3), 6 (1889), 9–102.

Miller, 1. " Abstract definitions of all the substitution groups whose degrees do not exceed seven ", *American J. of Math.*, 33 (1911), 363–372.

———, 2. " Groups generated by two operators of order three whose product is of order four ", *Bull. Amer. Math. Soc.*, 26 (1920), 361–369.

Sinkov, 1. " A set of defining relations for the simple group of order 1092 ", *Bull. Amer. Math. Soc.*, 41 (1935), 237–240.

Stott, 1. " Geometrical reduction of semiregular from regular polytopes and space fillings ", *Ver. der K. Akad. van Wet.* (Amsterdam) (1), 11.1 (1910).

Threlfall and Seifert, 1. " Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes ", *Math. Annalen*, 104 (1931), 1–70.

Todd and Coxeter, 1. " A practical method for enumerating co-sets of a finite abstract group ", *Proc. Edinburgh Math. Soc.* (2), 5 (1936), 26–34.

*Reference "Coxeter 4" is Chapter 3 of this book.