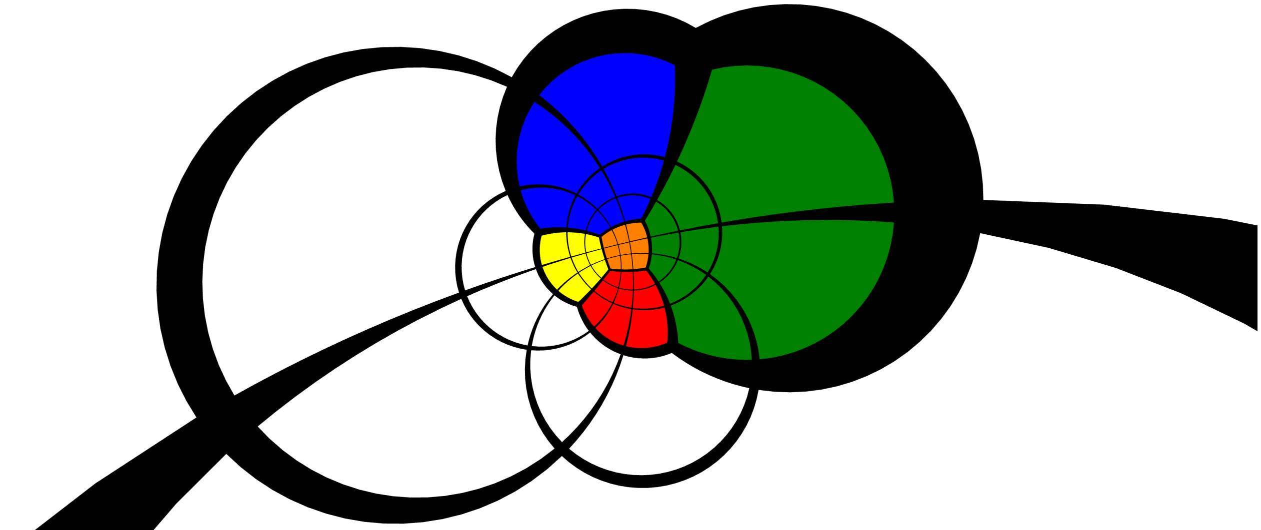


Abstracting Rubik's Cube

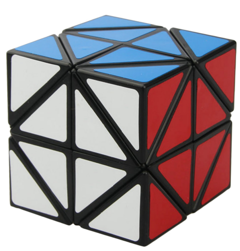
Roice Nelson

> "The art of doing mathematics consists in finding that special case that contains all the germs of generality" - David Hilbert

Over the past few decades, a growing group of "hypercubists" have been discovering analogues of Rubik's cube, traversing a wide range of mathematical ground. Solving puzzles is a core pastime, but this group is about much more. The explorations have been a microcosm of mathematical progress. Finding and studying natural analogues provides a rich way to approach varied topics in mathematics: geometry (higher-dimensional, non-Euclidean, projective), group theory, combinatorics, algorithms, topology, polytopes, tilings, honeycombs, and more. Elegance is a core principle in the quest.

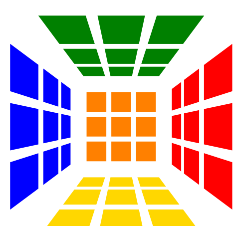
Those exposed to the twisty puzzle community know there are many properties of the classic *3x3x3* Rubik's Cube we can change to make new and interesting puzzles, for example by altering the shape or the twist centers as in Figure 1. The hypercubing group began by changing a more abstract property, namely the dimension. Don Hatch and Melinda Green wrote an exquisite working 4-dimensional *3x3x3x3* (or 3^4) analogue, which they called MagicCube4D. Using *dimensional analogy*, every property of this puzzle is upped a dimension. Faces, stickers, and twisting are 3D rather than 2D. Using a central $4D \rightarrow 3D$ projection, we see the hyperpuzzle as if you are looking into a box, with the nearest face hidden. Figure 2 puts the 3D and 4D puzzles side-by-side to emphasize the analogy.

(a) "Megaminx" uses a dodecahedral shape rather than a cube.

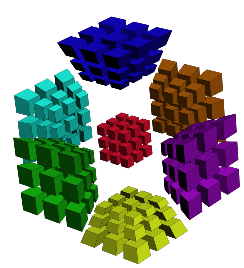


(b) The "Helicopter Cube" twists around edges instead of faces.

Figure 1: We begin abstracting Rubik's cube as soon as we change some property.



(a) The 3^3, projected so a 2-dimensional "flatlander" sees 5 of the 6 cube faces.



(b) The 3^4, projected so a 3-dimensional being sees 7 of the 8 hypercube faces.

Figure 2: Dimensional analogy and projection tricks can help us understand higher dimensional Rubik's Cubes.

The 3^3 Rubik's Cube has $6 \times 3^2 = 54$ stickers that can live in a mind-boggling 4.325×10^{19} possible states. The hypercubical 3^4 has $8 \times 3^3 = 216$ stickers and the number of possible puzzle positions explodes to an incomprehensible 1.756×10^{120}. Calculating this number is a challenge that will test your group theory mettle!

> "In that blessed region of Four Dimensions, shall we linger on the threshold of the Fifth, and not enter therein?" - Edwin Abbott, Flatland

The group didn't stop there. In 2006, a working 5-dimensional puzzle materialized with $10 \times 3^4 = 810$ hypercubical stickers and 7.017×10^{560} states, pushing the boundaries of visualization. The picture on the screen is effectively a shadow of a shadow of a shadow of the 5D object. Nonetheless, as of mid 2017, around seventy individuals have solved this puzzle. In June 2010, Andrey Astrelin stunned the group by using a creative visual approach to represent a 7-dimensional Rubik's Cube. Yes, it has been solved. Can you calculate the number of stickers on the 3^7? You may also enjoy using dimensional analogy to work out the properties of a 2-dimensional Rubik's Cube. What dimension are the stickers?

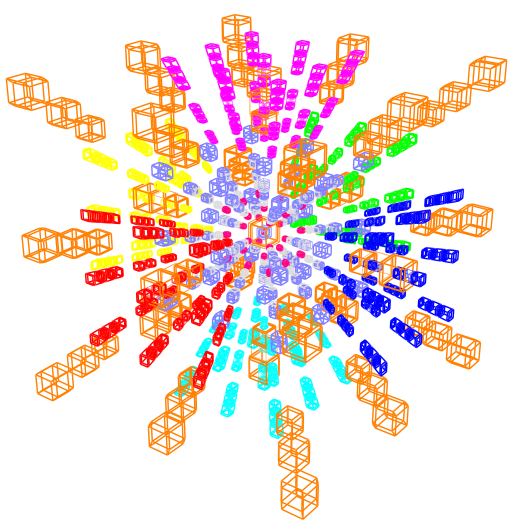


Figure 3: A shadow of a shadow of a shadow of the 3^5. Stickers are little hypercubes.

Of course we can play the same game of changing the shape in higher dimensions to yield a panoply of additional puzzles. There are 5 Platonic solids in 3 dimensions, but 6 perfectly regular shapes a dimension up, and you can attempt to solve twisty puzzle versions of all of them! Figure 4 shows one of the most beautiful in its pristine state.

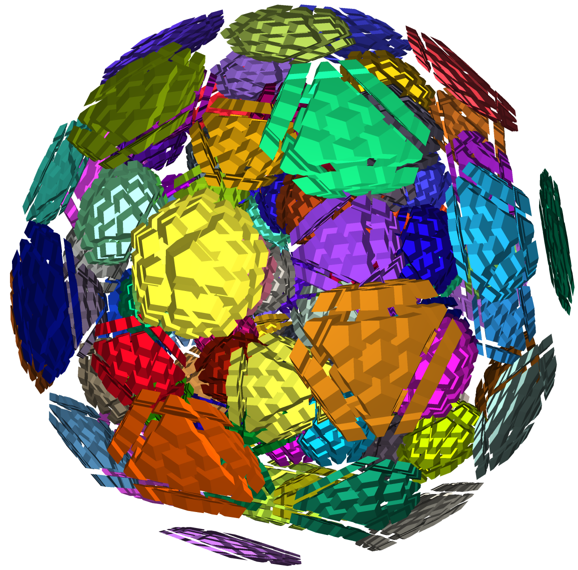


Figure 4: Magic120Cell, or the "4D Megaminx" has 120 dodecahedral faces. It derives from the *120-Cell*, one of 6 Platonic shapes in 4 dimensions.

Shapes in arbitrary dimensions are called *polytopes*, or *polychora* in 4 dimensions. In addition to the regular polychora, there are many *uniform* polychora and quite a few have been turned into twisty puzzles. Uniform polychora can break regularity in various ways. They may have multiple kinds of 3D faces or the faces may be composed of uniform (a.k.a. *Archimedean*) polyhedra.

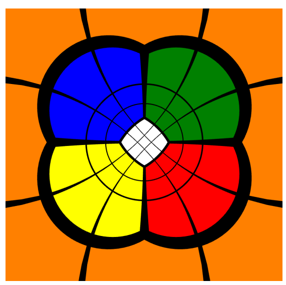
> "For God's sake, I beseech you, give it up. Fear it no less than sensual passions because it too may take all your time and deprive you of your health, peace of mind and happiness in life."

No, these were not desperate pleas to a hypercubist about excessive puzzling adventures. Such were the words of Farkas Bolyai to his son János, discouraging him from investigating Euclid's fifth postulate. János continued nonetheless, which led him into the wonderful world of hyperbolic geometry. We also did not heed the advice.

Let's use topology to abstract away a different property of Rubik's Cube - its cubeness. To do this, we project the cube faces radially outward onto a sphere. Mathematicians label the sphere S^2 because they consider it a 2-dimensional surface rather than a 3-dimensional object. Notice in Figure 5a that although the familiar cubeness is gone, all of the important combinatorial properties remain. Furthermore, what were 2-dimensional planar slices of the Rubik's cube are now 1-dimensional circles on the spherical surface. A twist simply rotates the portion of the surface inside one of these "twisting" circles.

In short, we are considering the Rubik's cube as a 2-dimensional tiling of the sphere by squares, sliced up by circles on the surface. Why? Because we can then consider other colored regular tilings and a huge number of new twisty puzzles become possible, some living in the world of hyperbolic geometry!

(a) Radially projected to the sphere, S^2.

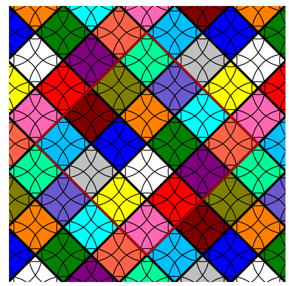


(b) *Stereographically* projected from the sphere to the complex plane.

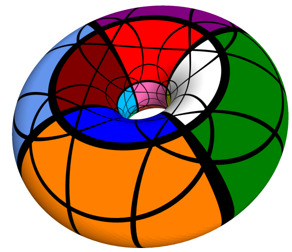
Figure 5: The Rubik's cube viewed as a 2-dimensional tiling on a surface.

In 2 dimensions, there are three geometries with constant curvature: spherical, Euclidean, and hyperbolic, and each can be tiled with regular polygons. These geometries correspond to whether the interior angles of a triangle sum to greater than, equal to, or less than 180 degrees, respectively. The *Schläfli symbol* efficiently encodes regular 2-dimensional tilings in all of these geometries with just two numbers, $\{p, q\}$. This denotes a tiling of p-gons in which q such polygons meet at each vertex. For example, $\{4, 3\}$ denotes the tiling of squares with three arranged around each vertex, i.e. the cube. The value of $(p-2)(q-2)$ determines the geometry: Euclidean when equal to 4, spherical when less, and hyperbolic when greater.

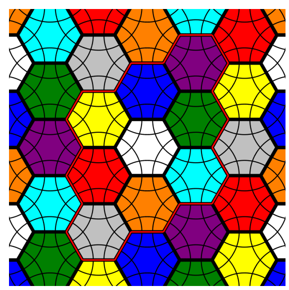
Euclidean geometry is the only one of the three that can live on the plane without any distortion. A lovely way to represent the others on the plane is via *conformal*, or angle preserving, maps. *Stereographic* projection is a conformal map for spherical geometry. Its analogue for hyperbolic geometry is the *Poincaré disk*, which squashes the infinite expanse of the hyperbolic plane into a unit disk. These models have many beautiful properties and the *isometries* (transformations which preserve length) of all 3 models can be described via a simple mathematical expression that



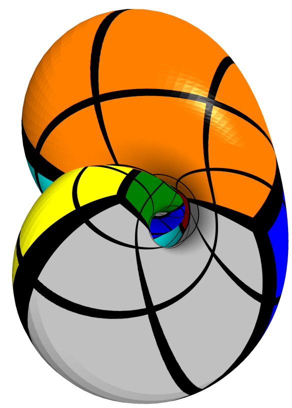
(a) Torus Rubik's cube on the Euclidean universal cover.



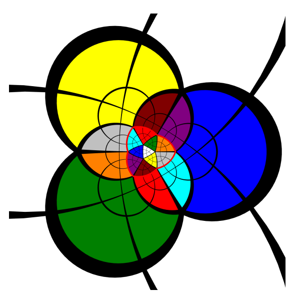
(b) Torus Rubik's cube mapped to the *Clifford* torus.



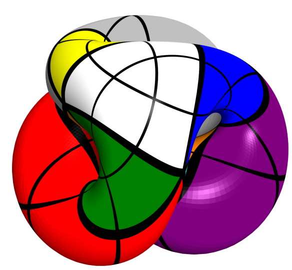
(c) Klein bottle Rubik's cube on the Euclidean universal cover.



(d) Klein bottle Rubik's cube mapped to a *Lawson* Klein bottle.



(e) Hemi-icosahedron (or real projective plane) Rubik's cube on the spherical universal cover.



(f) Hemi-icosahedron Rubik's cube mapped to the *Bryant–Kusner parametrization* of Boy's surface.

Figure 6: Example tiling analogues. Note that there are other tilings that can also map to the surfaces on the right.

acts on the complex plane: the *Möbius* transformations.

$$f(z) = \frac{az+b}{cz+d}$$

You may have noticed that we have with another problem to make puzzle analogues workable for Euclidean and hyperbolic tilings. Spherical tilings are finite, but tilings of the other two geometries go on forever. To overcome this final hurdle, we take a fundamental set of tiles and identify edges to be glued up into a *quotient* surface. This serves to make the infinite tilings into finite puzzles. Figure 6 show but a few examples. We can even glue up a subset of tiles on the sphere, as in Figure 6e.

One of the crown jewels of this abstraction is the *Klein Quartic* Rubik's cube, composed of 24 heptagons, three meeting at each vertex. It has "center", "edge", and "corner" pieces just like Rubik's cube. The *universal cover* is the $\{7,3\}$ hyperbolic tiling, and the quotient surface it is living on turns out to be a 3 holed torus. This results in some solution surprises; if you solve layer-by-layer as is common on the Rubik's cube, you'll find yourself left with **two** unsolved faces at the end instead of one.

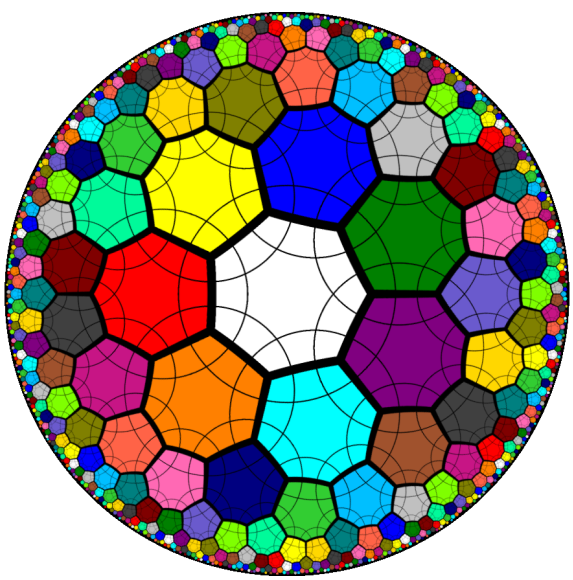


Figure 7: Klein Quartic Rubik's cube on the hyperbolic universal cover. The quotient surface is a 3 holed torus.

All of these puzzles and more are implemented in program called MagicTile. The puzzle count recently exceeded a thousand, with literally an infinite number of possibilities remaining.

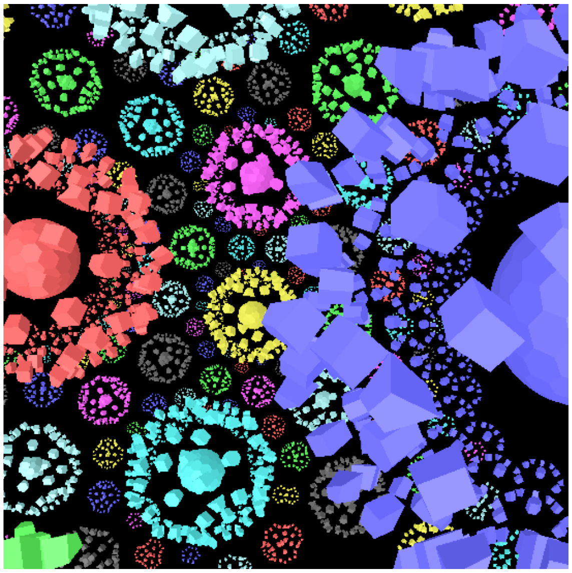
"We live on an island surrounded by a sea of ignorance. As our island of knowledge grows, so does the shore of our ignorance." - John Archibald Wheeler

There are quite a few intriguing analogues that I have not been able to describe here. Let me just mention two of my favorite abstractions, shown in Figure 8. The first is another astonishing set of puzzles by Andrey are based on the $\{6,3,3\}$ *honeycomb* in 3-dimensional hyperbolic space, $\mathbb{H}^3$. The faces are hexagonal $\{6,3\}$ tilings, with 3 faces meeting at each edge. Gluing via identifications serve to make the underlying honeycomb finite in two senses: the number of faces and the number of facets per face. If we take a step back and consider where we started, this puzzle has altered the dimension, the geometry, and the shape compared to the original Rubik's cube!

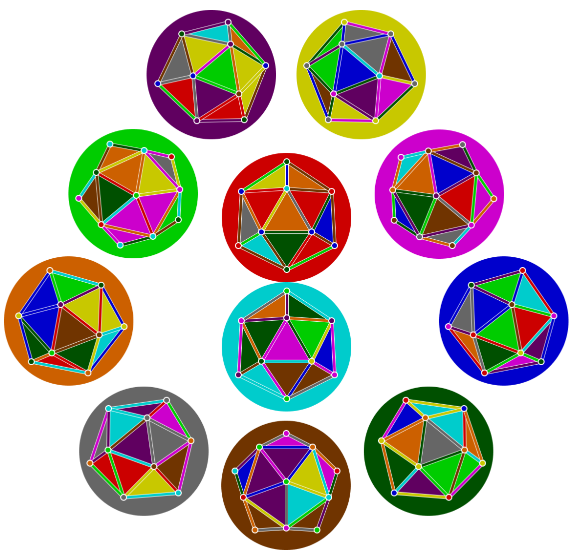
The second is a puzzle written by Nan Ma based on the 11-cell, an *abstract regular polytope* composed of eleven hemi-icosahedral cells. You might consider this a higher dimensional cousin of the Boy's surface puzzle we met earlier. The 11-cell can only live geometrically unwarped in ten dimensions, but Nan was able to preserve the combinatorics in his depiction.

With so many puzzles having been uncovered, one could be forgiven for suspecting there is not much more to do. On the contrary, there are arguably more avenues to approach new puzzles now than ten years ago. For example, there are no working puzzles in $\mathbb{H}^3$ composed of finite polyhedra. There are not yet puzzles for uniform tilings of euclidean or hyperbolic geometry, in 2 or 3 dimensions. Uniform tilings are not even completely classified, so further mathematics is required before some puzzles can be realized. Melinda has been developing a physical puzzle that is combinatorially equivalent to the 2^4. The idea of *fractal* puzzles have come up, but no one has yet been able to find a good analogue.

In addition to the search for puzzles, countless mathematical questions have been asked or are ripe for investigation. How many permutations do the various puzzles have? What checkerboard patterns are possible? Which n^d puzzles have the same number of stickers as pieces? How many ways can you color the faces of the 120-Cell puzzle? What is the nature of *God's number* for higher dimensional Rubik's cubes? The avenues seem limited only by our curiosity.



(a) Magic Hyperbolic Tile $\{6, 3, 3\}$. This is an in-space view of the puzzle in 3-dimensional hyperbolic space.



(b) Magic 11-Cell. Here we see the puzzle scrambled.

Figure 8: Two extremely exotic Rubik's cube abstractions.

Furthur Exploration

MagicCube4D website
Contains links to all the puzzles in this article and the hypercubing mailing list.

Burkard Polster (Mathologer) produced wonderful introductory videos to MagicCube4D and MagicTile.
Cracking the 4D Rubik's Cube with simple 3D tricks
Can you solve THE Klein Bottle Rubik's cube?

The following papers are freely available online:

Kamack, H. J., and T. R. Keane. ”The Rubik Tesseract.” (1982).

Stillwell, John. ”The Story of the 120-cell.” Notices of the AMS 48.1 (2001).

Séquin, Carlo H., Jaron Lanier, and UC CET. ”Hyperseeing the regular Hendecachoron.” Proc ISAMA (2007): 159-166.

Roice is a software developer with a passion for exploring mathematics through visualization. He enjoys spending time with his soul-mate Sarah and their three cats, and prefers traveling on two or fewer wheels.

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